CONNECTED MATHEMATICS® 3

Growing, Growing, Growing

Exponential Functions

Lappan, Phillips, Fey, Friel

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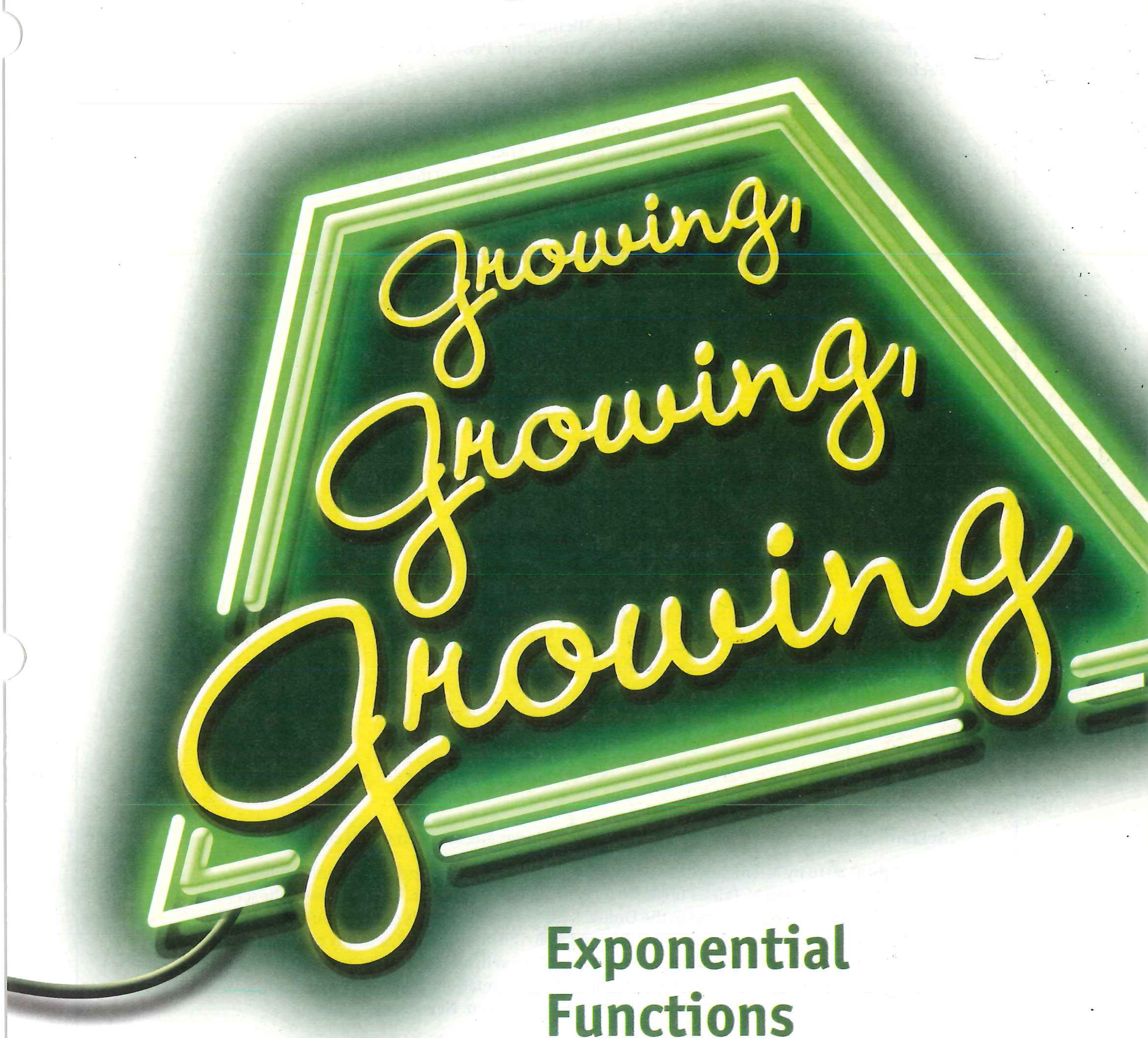

Exponential Functions

Glenda Lappan, Elizabeth Difanis Phillips,
James T. Fey, Susan N. Friel

Pearson
Boston, Massachusetts

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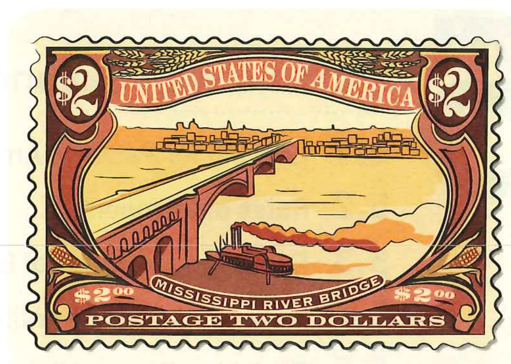
Growing, Growing, Growing

Exponential Functions

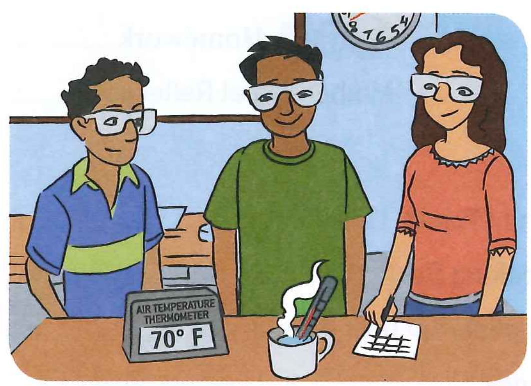
Looking Ahead

When the water hyacinth was introduced to Lake Victoria, it spread quickly over the lake's surface. At one point, the plant covered 769 square miles, and its area doubled every 15 days. **What** equation models this growth?

When Sam was in seventh grade, his aunt gave him a stamp worth $2,500. The value of the stamp increased by 6% each year for several years in a row. **What** was the value of Sam's stamp after four years?



What pattern of change would you expect to find in the temperature of a hot drink as time passes? **What** would a graph of the *(time, drink temperature)* data look like?



One of the most important uses of algebra is to model patterns of change. You are already familiar with linear patterns of change or linear functions. Linear patterns have constant differences and straight-line graphs. In a linear function, the y-value increases by a constant amount each time the x-value increases by 1.

In this Unit, you will study exponential patterns of change for exponential functions. Exponential growth patterns are fascinating because, although the values may change gradually at first, they eventually increase very rapidly. Patterns that decrease, or decay, exponentially may decrease quickly at first, but eventually they decrease very slowly.

Mathematical Highlights

Exponential Functions

In *Growing, Growing, Growing,* you will explore Exponential Functions, one of the most important types of nonlinear relationships.

The Investigations in this Unit will help you learn how to:

- Identify situations in which a quantity grows or decays exponentially
- Recognize the connections between the growth patterns in tables, graphs, and equations that represent exponential functions
- Construct equations to express the relationship between the variables in an exponential function in data tables, graphs, and problem situations
- Compare exponential and linear functions
- Develop and use rules for working with exponents, including scientific notation, to write and interpret equivalent expressions
- Solve problems about exponential growth and decay from a variety of different areas, including science and business

ASK YOURSELF

As you work on the Problems in this Unit, ask yourself questions about situations that involve nonlinear relationships such as:

How can I recognize whether the relationship between the variables is an exponential function?

What is the growth or decay factor?

What equation models the data in the table, graph, or problem situation?

What can I learn about this situation by studying a table or graph of the exponential function?

How can I answer questions about the problem situation by studying a table, graph, or equation that represents the exponential function?

Common Core State Standards

Mathematical Practices and Habits of Mind

In the *Connected Mathematics* curriculum you will develop an understanding of important mathematical ideas by solving problems and reflecting on the mathematics involved. Every day, you will use "habits of mind" to make sense of problems and apply what you learn to new situations. Some of these habits are described by the *Common Core State Standards for Mathematical Practices* (MP).

MP1 Make sense of problems and persevere in solving them.

When using mathematics to solve a problem, it helps to think carefully about

- data and other facts you are given and what additional information you need to solve the problem;
- strategies you have used to solve similar problems and whether you could solve a related simpler problem first;
- how you could express the problem with equations, diagrams, or graphs;
- whether your answer makes sense.

MP2 Reason abstractly and quantitatively.

When you are asked to solve a problem, it often helps to

- focus first on the key mathematical ideas;
- check that your answer makes sense in the problem setting;
- use what you know about the problem setting to guide your mathematical reasoning.

MP3 Construct viable arguments and critique the reasoning of others.

When you are asked to explain why a conjecture is correct, you can

- show some examples that fit the claim and explain why they fit;
- show how a new result follows logically from known facts and principles.

When you believe a mathematical claim is incorrect, you can

- show one or more counterexamples—cases that don't fit the claim;
- find steps in the argument that do not follow logically from prior claims.

MP4 Model with mathematics.

When you are asked to solve problems, it often helps to

- think carefully about the numbers or geometric shapes that are the most important factors in the problem, then ask yourself how those factors are related to each other;
- express data and relationships in the problem with tables, graphs, diagrams, or equations, and check your result to see if it makes sense.

MP5 Use appropriate tools strategically.

When working on mathematical questions, you should always

- decide which tools are most helpful for solving the problem and why;
- try a different tool when you get stuck.

MP6 Attend to precision.

In every mathematical exploration or problem-solving task, it is important to

- think carefully about the required accuracy of results; is a number estimate or geometric sketch good enough, or is a precise value or drawing needed?
- report your discoveries with clear and correct mathematical language that can be understood by those to whom you are speaking or writing.

MP7 Look for and make use of structure.

In mathematical explorations and problem solving, it is often helpful to

- look for patterns that show how data points, numbers, or geometric shapes are related to each other;
- use patterns to make predictions.

MP8 Look for and express regularity in repeated reasoning.

When results of a repeated calculation show a pattern, it helps to

- express that pattern as a general rule that can be used in similar cases;
- look for shortcuts that will make the calculation simpler in other cases.

You will use all of the Mathematical Practices in this Unit. Sometimes, when you look at a Problem, it is obvious which practice is most helpful. At other times, you will decide on a practice to use during class explorations and discussions. After completing each Problem, ask yourself:

- What mathematics have I learned by solving this Problem?
- What Mathematical Practices were helpful in learning this mathematics?

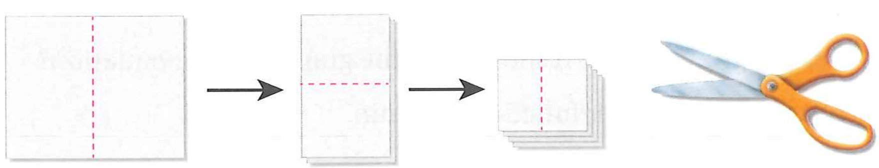
Investigation 1

Exponential Growth

In this Investigation, you will explore *exponential growth*. You will cut paper in half over and over to experience exponential growth. You will read a story about the land of Montarek. That story shows how exponential growth can be used. Finally, you will explore exponential patterns and compare them to linear growth patterns with tables, graphs, and equations.

1.1 Making Ballots
Introducing Exponential Functions

Chen is the secretary of the Student Government Association. He is making ballots for a meeting. Chen starts by cutting a sheet of paper in half. Then, he stacks the two pieces and cuts them in half again. With four pieces now, he stacks them and cuts them in half. By repeating this process, he makes smaller and smaller paper ballots.



Common Core State Standards

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

8.F.A.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

8.EE.A.3 Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other.

Also 8.EE.A.4, A-CED.A.1, A-CED.A.2, A-REI.B.3, F-IF.C.7, F-IF.C.7a, F-IF.C.7e, F-IF.C.9, F-BF.A.1, F-BF.A.1a, F-LE.A.1, F-LE.A.1a, F-LE.A.3, F-LE.B.5

After each cut, Chen counts the ballots and records the results in a table.

Number of Cuts	Number of Ballots
1	2
2	4
3	
4	
5	

He wants to predict the number of ballots after any number of cuts.

Describe the pattern of change. How many ballots are there after *n* cuts?

Problem 1.1

A **1.** Make a table to show the number of ballots after each of the first 5 cuts.

2. Look for a pattern in the way the number of ballots changes with each cut. Use your observations to extend your table to show the number of ballots for up to 10 cuts.

B **1.** Graph the data and write an equation that represents the relationship between the number of ballots and the number of cuts.

2. How does the growth pattern show up in the graph and the equation?

3. Is this relationship a linear function? Explain.

C **1.** Suppose Chen could make 20 cuts. How many ballots would he have? How many ballots would he have if he could make 40 cuts?

2. How many cuts would it take to make 500 ballots?

ACE Homework starts on page 14.

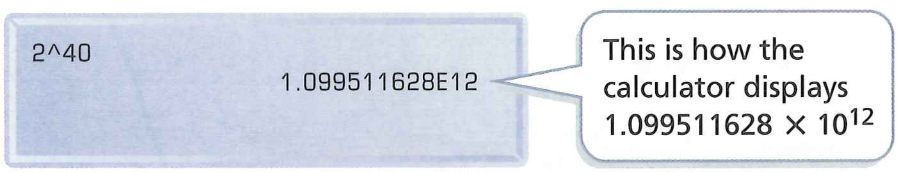
1.2 Requesting a Reward

Representing Exponential Functions

When you found the number of ballots after 10, 20, and 40 cuts, you may have multiplied long strings of 2s. Instead of writing long product strings of the same factor, you can use **exponential form,** such as 2^5. You can write $2 \times 2 \times 2 \times 2 \times 2$ as 2^5, which is read "2 to the fifth power."

In the expression 2^5, 5 is the **exponent** and 2 is the **base.** When you evaluate 2^5, you get $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. Since there are two ways to write 2^5, we call 32 the **standard form** and $2 \times 2 \times 2 \times 2 \times 2$ the **expanded form** of 2^5.

Stella used her calculator in Problem 1.1 to compute the number of ballots after 40 cuts. Calculators use shorthand for displaying very large numbers.



The number $1.099511628 \times 10^{12}$ is written in **scientific notation.**

This notation can be expanded as follows:

$$\begin{aligned} 1.099511628 \times 10^{12} &= 1.099511628 \times 1{,}000{,}000{,}000{,}000 \\ &= 1{,}099{,}511{,}628{,}000 \end{aligned}$$

The number 1,099,511,628,000 is the standard form for the number $1.099511628 \times 10^{12}$ written in scientific notation.

The calculator above has approximated 2^{40} as accurately as it can with the number of digits it can store. A number written in scientific notation must be in the form:

(a number greater than or equal to 1 but less than 10) $\times$ *(a power of 10)*

As you explore the king's dilemma below, you can use scientific notation to express large numbers.

One day in the ancient kingdom of Montarek, a peasant saved the life of the king's daughter. The king was so grateful he told the peasant she could have any reward she desired. The peasant, the kingdom's chess champion, made an unusual request:

Plan 1—The Peasant's Plan

"I would like you to place 1 ruba on the first square of my chessboard, 2 rubas on the second square, 4 on the third square, 8 on the fourth square, and so on. Continue this pattern until you have covered all 64 squares. Each square should have twice as many rubas as the previous square."

The king replied, "Rubas are the least valuable coin in the kingdom. Surely you can think of a better reward." But the peasant insisted, so the king agreed to her request.

- Did the peasant make a wise choice? Explain.

Problem 1.2

A 1. Make a table showing the number of rubas the king will place on squares 1 through 10 of the chessboard.

2. Graph the points (*number of the square, number of rubas*) for squares 1 to 10.

3. Write an equation for the relationship between the number of the square n and the number of rubas r.

B 1. How does the number of rubas change from one square to the next?

2. How does the pattern of change you observed in the table show up in the graph? How does it show up in the equation?

C 1. Which square will have 2^{30} rubas? Explain.

2. What is the first square on which the king will place at least one million rubas? How many rubas will be on this square?

3. Larissa uses a calculator to compute the number of rubas on a square. When is the first time the answer is displayed in scientific notation?

D Compare the growth pattern to the growth pattern in Problem 1.1.

ACE Homework starts on page 14.

1.3 Making a New Offer
Growth Factors

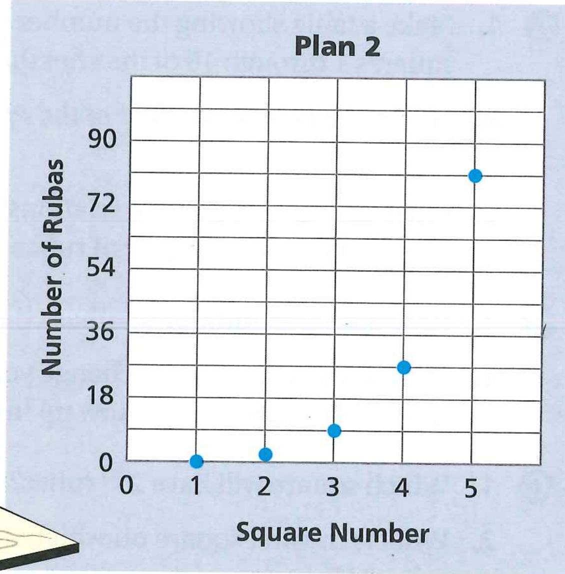
The patterns of change in the number of ballots in Problem 1.1 and in the number of rubas in Problem 1.2 show **exponential growth.** In each case, you can find the value for any cut or square by multiplying the previous value by a fixed number. This fixed number is called the **growth factor.** These relationships are called **exponential functions.** The number of the cut or square is the *independent variable.* The number of pieces of paper or rubas on the square is the *dependent variable.*

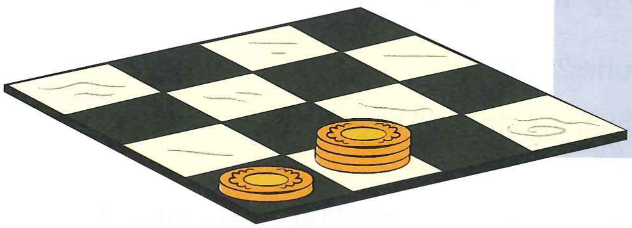
- What are the growth factors for Problems 1.1 and 1.2?

The king told the queen about the reward he promised the peasant. The queen said, "You have promised her more money than the entire royal treasury! You must convince her to accept a different reward."

Plan 2—The King's New Plan

After much thought, the king came up with Plan 2. He would make a new board with only 16 squares. Then he would place 1 ruba on the first square and 3 rubas on the second. He drew a graph to show the number of rubas on the first five squares. He would continue this pattern until all 16 squares were filled.





Plan 3—The Queen's Plan

The queen was unconvinced about the king's new plan. She devised Plan 3. Using a board with 12 squares, she would place 1 ruba on the first square. She would use the equation $r = 4^{n-1}$ to figure out how many rubas to put on each square. In the equation, r is the number of rubas on square n.

Problem 1.3

A **1.** In the table below, Plan 1 is the reward the peasant requested. Plan 2 is the king's new plan. Plan 3 is the queen's plan. Copy and extend the table to show the number of rubas on squares 1 to 10 for each plan.

Reward Plans

Square Number	Number of Rubas		
	Plan 1	Plan 2	Plan 3
1	1	1	1
2	2	3	4
3	4	■	■
4	■	■	■

Problem 1.3 continued

2. a. What are the independent and dependent variables in each plan?

   b. How are the patterns of change in the number of rubas under Plans 2 and 3 similar to Plan 1? How are they different from Plan 1?

3. Do the growth patterns for Plans 2 and 3 represent exponential functions? If so, what is the growth factor for each? Explain.

B 1. Write an equation for the relationship between the number of the square n and the number of rubas r for Plan 2.

2. Make a graph of Plan 3 for $n = 1$ to 10. How does your graph compare to the graphs for Plans 1 and 2?

3. How is the growth factor represented in the equations and graphs for Plans 2 and 3?

C The king's financial advisor said that either Plan 2 or Plan 3 would devastate the royal treasury. She proposed a fourth plan.

Plan 4—The Financial Advisor's Plan

The advisor proposed Plan 4. The king would put 20 rubas on the first square, 25 on the second, 30 on the third, and so on. He would increase the number of rubas by 5 for each square. He would continue this pattern until all 64 squares are covered.

1. Compare the growth pattern of Plan 4 to Plans 1, 2, and 3. Is the pattern in Plan 4 an exponential function? Explain.

2. Write an equation that represents the relationship in Plan 4.

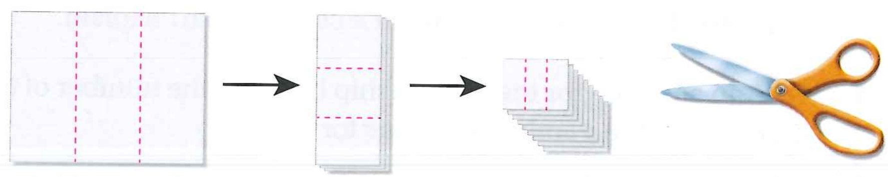
D For each plan, how many rubas are on the final square? List them from least to greatest.

ACE Homework starts on page 14.

Applications | Connections | Extensions

Applications

1. Cut a sheet of paper into thirds. Stack the three pieces and cut the stack into thirds. Stack all of the pieces and cut the stack into thirds again.



a. Copy and complete this table to show the number of ballots after repeating this process five times.

b. Suppose you continued this process. How many ballots would you have after 10 cuts? How many would you have after *n* cuts?

c. How many cuts would it take to make at least one million ballots?

Cutting Ballots

Cutting Processes	Number of Ballots
1	3
2	■
3	■
4	■
5	■

2. Chen, Lisa, Gabriel, and Artie each take a large piece of paper to make ballots. First, they cut the original piece of paper in half. Then, they cut each of those new pieces in half. Finally, they cut all of those pieces in half for a total of three cuts. They want to know how many ballots they will have without counting them. Each has a different conjecture. Who do you agree with? Explain.

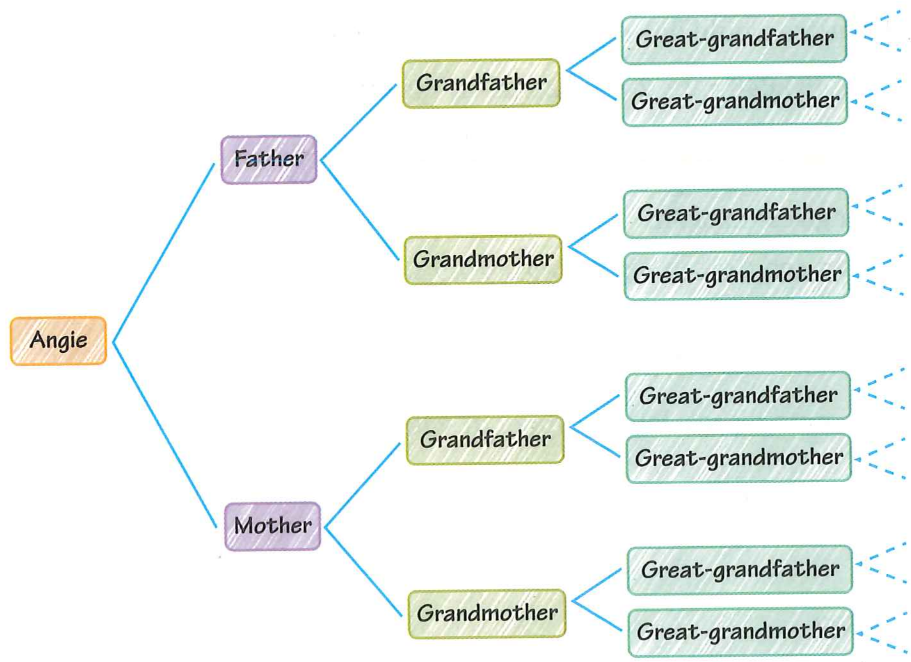
Chen's Conjecture The total number of ballots will be 2^{12} because as a group we made twelve total cuts.

Lisa's Conjecture The total number will be 8^4 because each person will have eight ballots, and there are four of us.

Gabriel's Conjecture The total number will be 4×2^3 because each person makes 2^3 ballots and there are four of us.

Artie's Conjecture The total number can't be determined using a formula. You will have to count them piece by piece.

3. Angie is studying her family's history. She discovers records of ancestors 12 generations back. She wonders how many ancestors she has from the past 12 generations. She starts to make a diagram to help her figure this out. The diagram soon becomes very complex.



a. Make a table and a graph showing the number of ancestors in each of the 12 generations.

b. Write an equation for the number of ancestors a in a given generation n.

c. What is the total number of ancestors in all 12 generations?

4. Sarah was working on Problem 1.2. She found that there will be 2,147,483,648 rubas on square 32.

a. How many rubas will be on square 33? How many will be on square 34? How many will be on square 35?

b. Which square would have the number of rubas shown here?

$$2{,}147{,}483{,}648 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$

c. Use your calculator to do the multiplication in part (b). Do you notice anything strange about the answer your calculator gives? Explain.

d. Write $2{,}147{,}483{,}648 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ in scientific notation.

e. Write the numbers 2^{10}, 2^{20}, 2^{30}, 2^{40}, and 2^{50} in scientific notation.

f. Explain how to write a large number in scientific notation.

For Exercises 5–7, write each number in scientific notation.

5. 100,000,000

6. 29,678,900,522

7. 11,950,500,000,000

For Exercises 8–10, write each number in standard form.

8. 6.43999001×10^8

9. 8.89234×10^5

10. $3.4348567000 \times 10^{10}$

11. What is the largest whole-number value of n that your calculator will display in standard notation?

a. 3^n

b. π^n

c. 12^n

d. 237^n

12. What is the smallest value of n that your calculator will display in scientific notation?

a. 10^n

b. 100^n

c. $1000n^n$

13. Many single-celled organisms reproduce by dividing into two identical cells.

Suppose an amoeba (uh MEE buh) splits into two amoebas every half hour.

a. A biologist starts an experiment with one amoeba. Make a table showing the number of amoebas she would have at the end of each hour over an 8-hour period.

b. Write an equation for the number of amoebas a after t hours. Which variable is the independent variable? Dependent variable?

c. How many hours will it take for the number of amoebas to reach one million?

d. Make a graph of the data (*time, amoebas*) from part (a).

e. What similarities do you notice in the pattern of change for the number of amoebas and the patterns of change for other situations in this Investigation? What differences do you notice?

14. Zak's uncle wants to donate money to Zak's school. He suggests three possible plans. Look for a pattern in each plan.

Plan 1 He will continue the pattern in this table until day 12.

School Donations

Day	1	2	3	4
Donation	$1	$2	$4	$8

Plan 2 He will continue the pattern in this table until day 10.

School Donations

Day	1	2	3	4
Donation	$1	$3	$9	$27

Plan 3 He will continue the pattern in this table until day 7.

School Donations

Day	1	2	3	4
Donation	$1	$4	$16	$64

a. Copy and extend each table to show how much money the school would receive each day.

b. For each plan, write an equation for the relationship between the day number n and the number of dollars donated d.

c. Are any of the relationships in Plans 1, 2, or 3 exponential functions? Explain.

d. Which plan would give the school the greatest total amount of money?

15. Carmelita is planning to swim in a charity swim-a-thon. Several relatives said they would sponsor her.

I will give you $1 if you swim 1 lap, $3 if you swim 2 laps, $5 if you swim 3 laps, $7 if you swim 4 laps, and so on.—**Grandmother**

I will give you $1 if you swim 1 lap, $3 if you swim 2 laps, $9 if you swim 3 laps, $27 if you swim 4 laps, and so on.—**Father**

I will give you $2 if you swim 1 lap, $3.50 if you swim 2 laps, $5 if you swim 3 laps, $6.50 if you swim 4 laps, and so on.—**Aunt Josie**

I will give you $1 if you swim 1 lap, $2 if you swim 2 laps, $4 if you swim 3 laps, $8 if you swim 4 laps, and so on.—**Uncle Sebastian**

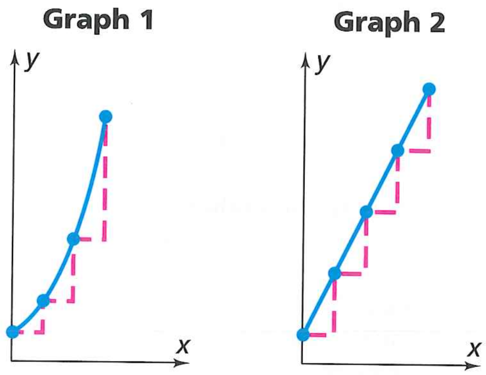
WOW! Thanks everyone for your support!—**Carmelita**

a. Decide whether each donation pattern is an *exponential function*, *linear function*, or *neither*.

b. For each relative, write an equation for the total donation d if Carmelita swims n laps. Which variable is the independent variable? Dependent variable?

c. For each plan, tell how much money Carmelita will raise if she swims 20 laps.

16. The graphs below represent the equations $y = 2^x$ and $y = 2x + 1$.



a. Tell which equation each graph represents. Explain your reasoning.

b. The dashed segments show the vertical and horizontal change between points at equal x intervals. For each graph, compare the vertical and horizontal change between pairs of points. What do you notice?

c. Does either equation represent an exponential function? A linear function? Explain.

For Exercises 17–21, study the pattern in each table.

a. Tell whether the relationship between x and y is a *linear function, exponential function,* or *neither*. Explain your reasoning.

b. If the relationship is a linear or exponential, give its equation.

17.

x	0	1	2	3	4	5
y	10	12.5	15	17.5	20	22.5

18.

x	0	1	2	3	4
y	1	6	36	216	1,296

19.

x	0	1	2	3	4	5	6	7	8
y	1	5	3	7	5	8	6	10	8

20.

x	0	1	2	3	4	5	6	7	8
y	2	4	8	16	32	64	128	256	512

21.

x	0	1	2	3	4	5
y	0	1	4	9	16	25

Connections

For Exercises 22–24, write each expression in exponential form.

22. $2 \times 2 \times 2 \times 2$

23. $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

24. $2.5 \times 2.5 \times 2.5 \times 2.5 \times 2.5$

For Exercises 25–27, write each expression in standard form.

25. 2^{10} **26.** 10^2 **27.** 3^9

28. You know that $5^2 = 25$. How can you use this fact to evaluate 5^4?

29. The standard form for 5^{10} is 9,765,625. How can you use this fact to evaluate 5^{11}?

30. **Multiple Choice** Which expression is equal to one million?

A. 10^6 **B.** 6^{10} **C.** 100^2 **D.** 2^{100}

31. Use exponents to write an expression for one billion (1,000,000,000).

For Exercises 32–34, decide whether each number is more or less than one million *without using a calculator* or multiplying. Explain how you found your answer. Use a calculator to check your answer.

32. 9^6 **33.** 3^{10} **34.** 11^6

For Exercises 35–40, write the number in exponential form using 2, 3, 4, or 5 for the base.

35. 125 **36.** 64 **37.** 81

38. 3,125 **39.** 1,024 **40.** 4,096

41. Refer to Problem 1.1. Suppose 250 sheets of paper is 1 inch high.

a. How high would the stack of ballots be after 20 cuts? After 30 cuts?

b. How many cuts would it take to make a stack 1 foot high?

c. The average distance from Earth to the moon is about 240,000 miles. Which (if any) of the stacks in part (a) would reach the moon?

42. In Problem 1.2, suppose a Montarek ruba has the value of a modern U.S. penny. What are the dollar values of the rubas on squares 10, 20, 30, 40, 50, and 60?

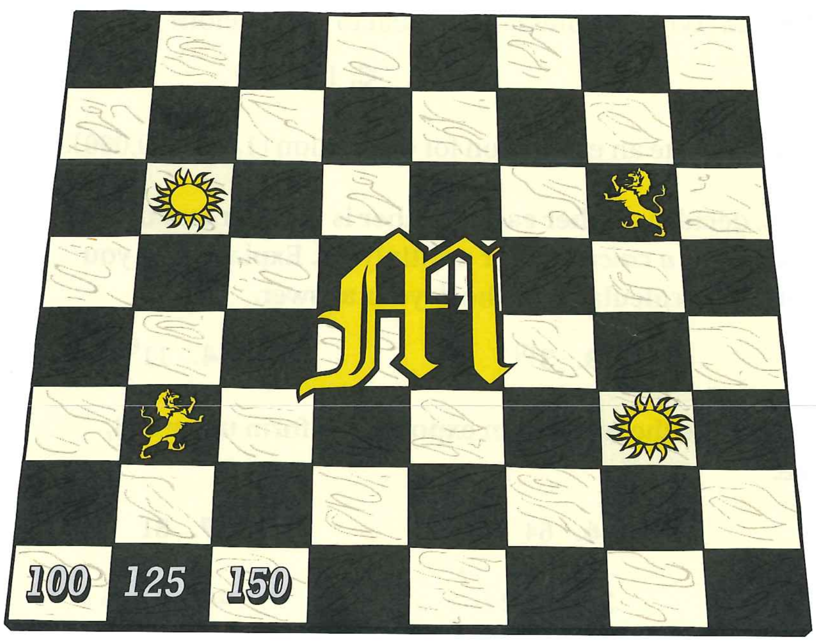
43. A ruba has the same thickness as a modern U.S. penny (about 0.06 inch). Suppose the king had been able to reward the peasant by using Plan 1 (doubling the number of rubas in each square). What would be the height of the stack of rubas on square 64?

44. One of the king's advisors suggested another plan. Put 100 rubas on the first square of a chessboard, 125 on the second square, 150 on the third square, and so on, increasing the number of rubas by 25 for each square.

a. Write an equation for the numbers of rubas r on square n. Explain the meanings of the numbers and variables in your equation.

b. Describe the graph of this plan.

c. What is the total number of rubas on the first 10 squares? The first 20 squares?



For Exercises 45–47, find the slope and y-intercept of the graph of each equation.

45. $y = 3x - 10$

46. $y = 1.5 - 5.6x$

47. $y = 15 + \frac{2}{5}x$

48. Write an equation whose line is less steep than the line represented by $y = 15 + \frac{2}{5}x$.

Extensions

49. Consider the two equations below.

Equation 1	**Equation 2**
$r = 3^n - 1$	$r = 3^{n-1}$

a. For each equation, find r when n is 2.

b. For each equation, find r when n is 10.

c. Explain why the equations give different values of r for the same value of n.

d. Do either of these equations represent an exponential function? Explain why.

50. The table below represents the number of ballots made by repeatedly cutting a sheet of paper in half.

Cutting Ballots

Number of Cuts	Number of Ballots
1	2
2	4
3	8
4	16

a. Write an equation for the pattern in the table.

b. Use your equation and the table to determine the value of 2^0.

c. What do you think b^0 should equal for any number b? For example, do you think 6^0 and 23^0 should equal? Explain.

51. The king tried to figure out the total number of rubas the peasant would receive under Plan 1. He noticed an interesting pattern.

a. Extend and complete this table for the first 10 squares.

Reward Plan 1

Square	Number of Rubas on Square	Total Number of Rubas
1	1	1
2	2	3
3	4	7
4	■	■

b. Describe the pattern of growth in the total number of rubas as the number of the square increases. Do either of these relationships represent an exponential function? Explain.

c. Write an equation for the relationship between the number of the square n and the total number of rubas on the board t.

d. When the total number of rubas reaches 1,000,000, how many squares will have rubas?

e. Suppose the king had been able to give the peasant the reward she requested. How many rubas would she have received?

52. Refer to Plans 1–4 in Problem 1.3.

a. Which plan should the king choose? Explain.

b. Which plan should the peasant choose? Explain.

c. Write an ending to the story of the king and the peasant.

Mathematical Reflections 1

In this Investigation, you explored situations in which the relationship between the two variables represented exponential functions. You saw how you could recognize patterns of exponential growth in tables, graphs, and equations.

Think about your answers to these questions. Discuss your ideas with other students and your teacher. Then write a summary of your findings in your notebook.

1. **Describe** an exponential growth pattern. Include key properties such as growth factors.
2. **How** are exponential functions similar to and different from the linear functions you worked with in earlier Units?

Common Core Mathematical Practices

As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems. Think back over your work, the ways you thought about the Problems, and how you used Mathematical Practices.

Tori described her thoughts in the following way:

We wrote the equation, $r = \frac{1}{2}2^n$ to represent the relationship between the number of rubas, r on square n in Problem 1.2. As the number of squares increase by 1, the number of rubas doubles.

We went backwards in the table to find the number of rubas on square 0. To find the number of rubas on square 0, we divided the number of rubas on square 1 by 2. One divided by two is $\frac{1}{2}$. $\frac{1}{2}$ is the y-intercept.

If you start with square 0, you get the number of rubas on the next square by multiplying the number of rubas on square 0 by 2. This process is repeated for the next square, etc.

So, on square n, you multiply $\frac{1}{2}$ by 2, n times. $\frac{1}{2} \times 2 \times 2 \times \ldots \times 2$ or $\frac{1}{2}(2^n)$.

Common Core Standards for Mathematical Practice

MP3 Construct viable arguments and critique the reasoning of others

- What other Mathematical Practices can you identify in Tori's reasoning?
- Describe a Mathematical Practice that you and your classmates used to solve a different Problem in this Investigation.

Investigation 2

Examining Growth Patterns

In Investigation 1, you learned to recognize exponential growth patterns. Now you are ready to take a closer look at the tables, graphs, and equations that represent exponential functions. You will explore this question:

- How do the starting value and growth factor show up in the table, graph, and equation that represent an exponential function?

For example, students at West Junior High wrote two equations to represent the reward in Plan 1 of Problem 1.2. Some students wrote $r = 2^{n-1}$ and others wrote $r = \frac{1}{2}\left(2^{n}\right)$. In both equations, r represents the number of rubas on square n.

- Are both equations correct? Explain.
- What is the value of r in both equations if $n = 1$? Does this make sense?
- What is the y-intercept for the graph of these equations?
- Do you think there is any value for n that will result in more than one value for r?

Common Core State Standards

8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

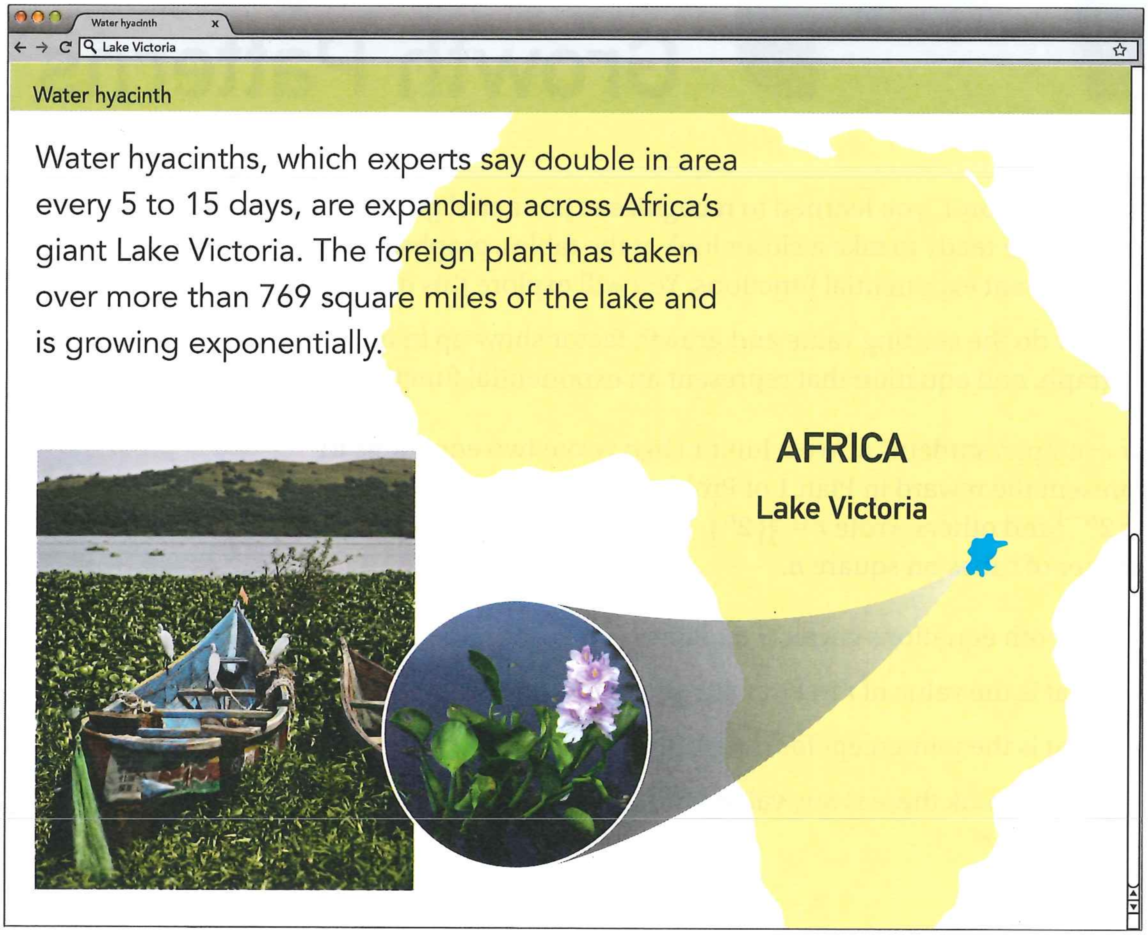
8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph. Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Also N-Q.A.1, N-Q.A.2, A-SSE.A.1, A-SSE.A.1a, A-CED.A.1, A-CED.A.2, F-IF.C.7e, F-BF.A.1, F-BF.A.1a, F-LE.A.1, F-LE.A.1a, F-LE.A.2, F-LE.B.5

2.1 Killer Plant Strikes Lake Victoria

y-Intercepts Other Than 1

Exponential functions occur in many real-life situations. For example, consider this story:



Little progress has been made to reverse the effects of the water hyacinths. Plants like the water hyacinth that grow and spread rapidly can affect native plants and fish. This in turn can affect the livelihood of fishermen. It can also impede rescue operations in case of a water disaster. To understand how such plants grow, you will look at a similar situation.

Problem 2.1

Ghost Lake is a popular site for fishermen, campers, and boaters. In recent years, a certain water plant has been growing on the lake at an alarming rate. The surface area of Ghost Lake is 25,000,000 square feet. At present, the plant covers 1,000 square feet of the lake. The Department of Natural Resources estimates that the area covered by the water plant is doubling every month.

A 1. Write an equation that represents the growth pattern of the plant.

2. Explain what information the variables and numbers in your equation represent.

3. Compare this equation to the equations in Investigation 1.

B 1. Make a graph of the equation.

2. How does this graph compare to the graphs of the exponential functions in Investigation 1?

3. Recall that a function is a relationship between two variables where, for each value of the independent variable, there is exactly one corresponding value of the dependent variable. Is the plant growth relationship a function? Justify your answer using a table, graph, or equation.

C 1. How much of the lake's surface will be covered at the end of a year by the plant?

2. How many months will it take for the plant to completely cover the surface of the lake?

ACE Homework starts on page 32.

2.2 Growing Mold

Interpreting Equations for Exponential Functions

Mold can spread rapidly. For example, the area covered by mold on a loaf of bread that is left out in warm weather grows exponentially.

Problem 2.2

Students at Magnolia Middle School conducted an experiment. They put a mixture of chicken bouillon (BOOL yahn), gelatin, and water in a shallow pan. Then they left it out to mold. Each day, the students recorded the area of the mold in square millimeters.

The students wrote the equation $m = 50(3^d)$ to model the growth of the mold. In this equation, m is the area of the mold in square millimeters after d days.

A For each part, answer the question and explain your reasoning.

1. What is the area of the mold at the start of the experiment?
2. What is the growth factor?
3. What is the area of the mold after 5 days?
4. On which day will the area of the mold reach 6,400 mm^2?

B An equation that represents an exponential function can be written in the form $y = a(b^x)$ where a and b are constant values.

1. What is the value of b in the mold equation? What does this value represent? Does this make sense in this situation? Explain.
2. What is the value of a in the mold equation? What does this value represent?

ACE Homework starts on page 32.

2.3 Studying Snake Populations

Interpreting Graphs of Exponential Functions

Garter snakes were introduced to a new area 4 years ago. The population is growing exponentially. The relationship between the number of snakes and the year is modeled with an exponential function.

Problem 2.3

A The graph shows the growth of the garter snake population.

1. Find the snake population for years 2, 3, and 4.
2. Use the pattern in your answers from part (1) to estimate the population in Year 1. Explain your reasoning.
3. Explain how you can find the y-intercept for the graph.

B Explain how to find the growth factor for the population.

C Write an equation relating time t in years and population p. Explain what information the numbers in the equation represent.

D In what year is the population likely to reach 1,500?

E Amy and Chuck were discussing whether this relationship represented an exponential function. Who is correct? Explain why.

Amy's claim It is not a function. When the independent variable is 4, it looks like there is more than one dependent value associated with it.

OR

Chuck's claim It is a function. The scale used for the graph makes it difficult to read the values when the independent variable is 4.

ACE Homework starts on page 32.

ACE Applications | Connections | Extensions

Applications

1. If you don't brush your teeth regularly, it won't take long for large colonies of bacteria to grow in your mouth. Suppose a single bacterium lands on your tooth and starts multiplying by a factor of 4 every hour.

   a. Write an equation that describes the number of bacteria b in the new colony after n hours.

   b. How many bacteria will be in the colony after 7 hours?

   c. How many bacteria will be in the colony after 8 hours? Explain how you can find this answer by using the answer from part (b) instead of the equation.

   d. After how many hours will there be at least 1,000,000 bacteria in the colony?

   e. Suppose that, instead of 1 bacterium, 50 bacteria land in your mouth. Write an equation that describes the number of bacteria b in this colony after n hours.

   f. Under the conditions of part (e), there will be 3,276,800 bacteria in this new colony after 8 hours. How many bacteria will there be after 9 hours and after 10 hours? Explain how you can find these answers without going back to the equation from part (e).

2. Loon Lake has a "killer plant" problem similar to Ghost Lake in Problem 2.1. Currently, 5,000 square feet of the lake is covered with the plant. The area covered is growing by a factor of 1.5 each year.

   a. Copy and complete the table to show the area covered by the plant for the next 5 years.

   b. The surface area of the lake is approximately 200,000 square feet. How long will it take before the lake is completely covered?

Growth of Loon Lake Plant

Year	Area Covered (sq. ft)
0	5,000
1	■
2	■
3	■
4	■
5	■

3. Leaping Liang just signed a contract with a women's basketball team. The contract guarantees her \$20,000 the first year, \$40,000 the second year, \$80,000 the third year, \$160,000 the fourth year, and so on, for 10 years.

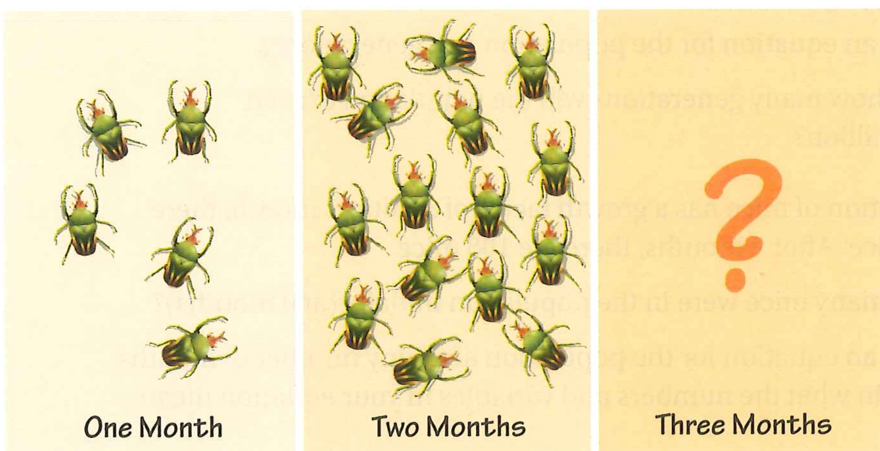
   a. Make a table showing Liang's salary for each year of this contract.

   b. What is the total amount Liang will earn over the 10 years?

   c. Does the relationship between the number of years and salary represent an exponential function? Explain.

   d. Write an equation for Liang's salary s for any year n of her contract.

4. As a biology project, Talisha is studying the growth of a beetle population. She starts her experiment with 5 beetles. The next month she counts 15 beetles.



   a. Suppose the beetle population is growing linearly. How many beetles can Talisha expect to find after 2, 3, and 4 months?

   b. Suppose the beetle population is growing exponentially. How many beetles can Talisha expect to find after 2, 3, and 4 months?

   c. Write an equation for the number of beetles b after m months if the beetle population is growing linearly. Explain what information the variables and numbers represent.

   d. Write an equation for the number of beetles b after m months if the beetle population is growing exponentially. Explain what information the variables and numbers represent.

   e. How long will it take the beetle population to reach 200 if it is growing linearly?

   f. How long will it take the beetle population to reach 200 if it is growing exponentially?

5. Fruit flies are often used in genetic experiments because they reproduce very quickly. In 12 days, a pair of fruit flies can mature and produce a new generation. The table below shows the number of fruit flies in three generations of a laboratory colony.

Growth of Fruit-Fly Population

Generations	0	1	2	3
Number of Fruit Flies	2	120	7,200	432,000

a. Does this data represent an exponential function? If so, what is the growth factor for this fruit-fly population? Explain how you found your answers.

b. Suppose this growth pattern continues. How many fruit flies will be in the fifth generation?

c. Write an equation for the population p of generation g.

d. After how many generations will the population exceed one million?

6. A population of mice has a growth factor of 3. After 1 month, there are 36 mice. After 2 months, there are 108 mice.

a. How many mice were in the population initially (at 0 months)?

b. Write an equation for the population after any number of months. Explain what the numbers and variables in your equation mean.

7. Fido did not have fleas when his owners took him to the kennel. The number of fleas on Fido after he returned from the kennel grew according to the equation $f = 8(3^n)$, where f is the number of fleas and n is the number of weeks since he returned from the kennel. (Fido left the kennel at week 0.)

a. How many fleas did Fido pick up at the kennel?

b. Is the relationship represented by the equation an exponential function? If so, what is the growth factor for the number of fleas?

c. How many fleas will Fido have after 10 weeks if they are untreated?

8. Consider the equation $y = 150(2^x)$.

   a. Make a table of x and y-values for whole-number x-values from 0 to 5.

   b. What do the numbers 150 and 2 in the equation tell you about the relationship between the variables x and y?

For Exercises 9–12, find the growth factor and the y-intercept of the equation's graph.

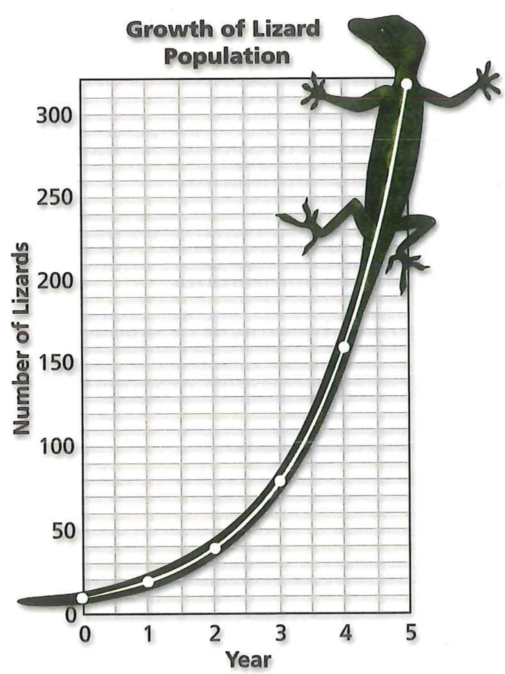
9. $y = 300(3^x)$

10. $y = 300(3)^x$

11. $y = 6{,}500(2)^x$

12. $y = 2(7)^x$

13. The following graph represents the population growth of a certain kind of lizard.



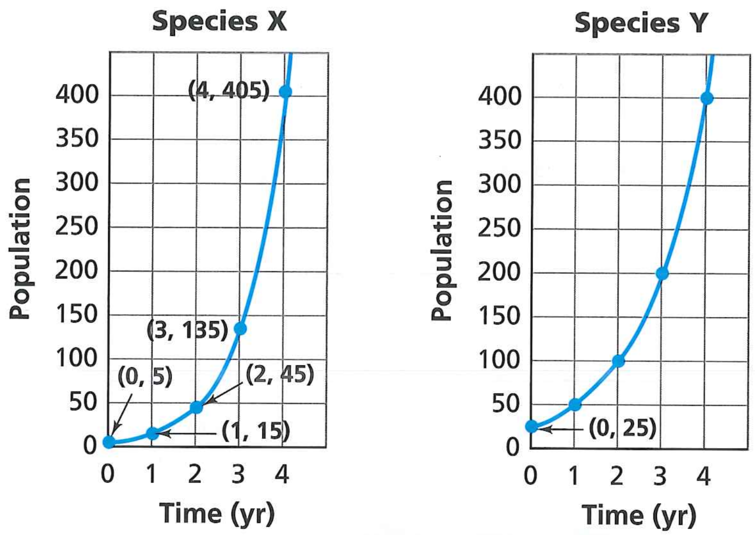
   a. What information does the point (2, 40) on the graph tell you?

   b. What information does the point (1, 20) on the graph tell you?

   c. When will the population exceed 100 lizards?

   d. Explain how you can use the graph to find the growth factor for the population.

14. The following graphs show the population growth for two species. Each graph represents an exponential function.



a. Find the growth factors for the two species.

b. What is the y-intercept for the graph of Species X? Explain what this y-intercept tells you about the population.

c. What is the y-intercept for the graph of Species Y? Explain what this y-intercept tells you about the population.

d. Write an equation that describes the growth of Species X.

e. Write an equation that describes the growth of Species Y.

f. For which equation is (5, 1215) a solution?

Connections

15. Multiple Choice Choose the answer that best approximates 3^{20} in scientific notation.

A. 3.5×10^{-9} **B.** 8×10^{3} **C.** 3×10^{9} **D.** 3.5×10^{9}

16. Multiple Choice Choose the answer that is closest to 2.575×10^{6}.

F. 21^{8} **G.** 12^{6} **H.** 6^{12} **J.** 11^{9}

17. Approximate 5^{11} in scientific notation.

For Exercises 18–20, decide whether each number is less than or greater than one million without using a calculator. Explain.

18. 3^6

19. 9^5

20. 12^6

For Exercises 21–23, write the prime factorization of each number using exponents. Recall the prime factorization of 54 is $3 \times 3 \times 3 \times 2$. This can be written using exponents as $3^3 \times 2$.

21. 45

22. 144

23. 2,024

24. Consider the two equations below.

Equation 1	Equation 2
$y = 10 - 5x$	$y = (10)5^x$

a. What is the y-intercept of each equation?

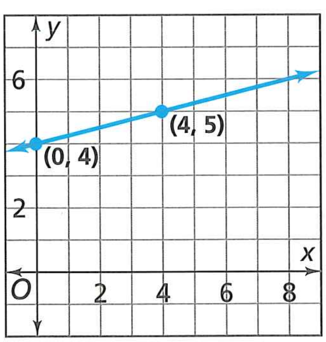
b. For each equation, explain how you could use a table to find how the y-values change as the x-values increase. Describe the change.

c. Explain how you could use the equations to find how the y-values change as the x-values increase.

d. For each equation, explain how you could use a graph to find how the y-values change as the x-values increase.

For Exercises 25–28, write an equation for each line. Identify the slope and y-intercept.

25.



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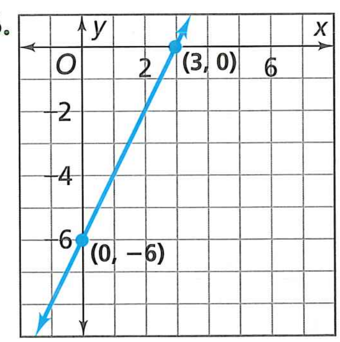
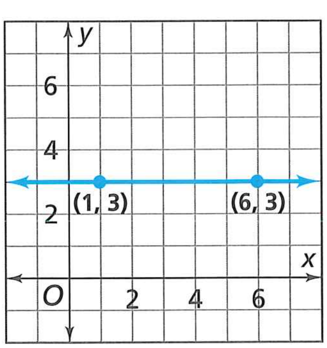
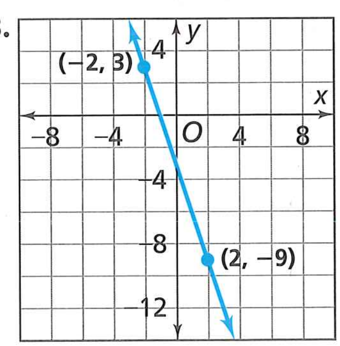


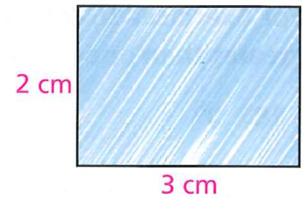
27.



28.



29. Maria enlarges a 2-cm-by-3-cm rectangle by a factor of 2 to get a 4-cm-by-6-cm rectangle. She then enlarges the 4-cm-by-6-cm rectangle by a factor of 2. She continues this process, enlarging each new rectangle by a factor of 2.



a. Copy and complete the table to show the dimensions, perimeter, and area of the rectangle after each enlargement.

Rectangle Changes

Enlargement	Dimensions (cm)	Perimeter (cm)	Area (cm^2)
0 (original)	2 by 3	■	■
1	4 by 6	■	■
2	■	■	■
3	■	■	■
4	■	■	■
5	■	■	■

b. Is the pattern of growth for the perimeter linear, exponential, or neither? Explain.

c. Does the pattern of growth for the area represent a linear function, exponential function, or neither? Explain.

d. Write an equation for the perimeter P after n enlargements.

e. Write an equation for the area A after n enlargements.

f. How would your answers to parts (a)–(e) change if the copier were set to enlarge by a factor of 3?

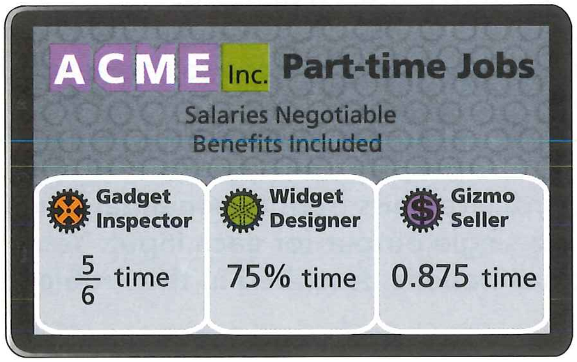
For Exercises 30 and 31, Kele enlarged the figure below by a scale factor of 2. Ahmad enlarged the figure 250%.

30. Who made the larger image?

31. Multiple Choice Which factor would give an image between Ahmad's image and Kele's image in size?

A. $\frac{2}{5}$ **B.** $\frac{3}{5}$ **C.** $\frac{9}{4}$ **D.** $\frac{10}{4}$

32. Companies sometimes describe part-time jobs by comparing them to full-time jobs. For example, a job that requires working half the number of hours of a full-time job is described as a $\frac{1}{2}$-time job or a 50%-time job. ACME, Inc. has three part-time job openings.



Order these jobs from the most time to the the least time.

Extensions

33. **a.** Make a table and a graph for the equation $y = 1^x$.

b. How are the patterns in the table and the graph of $y = 1^x$ similar to patterns you have observed for other exponential and linear functions? How are they different?

34. If you know that a graph represents an exponential function, you can find the equation for the function from two points on its graph. Find the equation of the exponential function whose graph passes through each pair of points. Explain.

a. (1, 6) and (2, 12) **b.** (2, 90) and (3, 270)

35. Leaping Liang plays basketball. A team promised her \$1 million a year for the next 25 years. The same team offered Dribbling Dinara \$1 the first year, \$2 the second year, \$4 the third year, \$8 the fourth year, and so on, for 25 years.

a. Suppose Liang and Dinara each accept the offers and play for 20 years. At the end of 20 years, who receives more money?

b. Tell which player will receive more after 21 years, 22 years, 23 years, and 25 years.

c. Do either of the two plans represent an exponential function? Explain.

Mathematical Reflections 2

In this Investigation, you studied quantities that grew exponentially. These patterns of growth represent exponential functions. You looked at how the values changed from one stage to the next, and you wrote equations to represent the relationship and used them to find the value of a quantity at any stage of growth.

You also graphed coordinate pairs from exponential functions and saw that there is a single output for each input. You sketched graphs from situations described in the Problems and analyzed graphs.

Think about your answers to these questions. Discuss your ideas with other students and your teacher. Then write a summary of your findings in your notebook.

1. **How** can you use a table, a graph, and an equation that represent an exponential function to find the *y*-intercept and growth factor for the function? Explain.
2. **How** can you use the *y*-intercept and growth factor to write an equation that represents an exponential function? Explain.
3. **How** would you change your answers to Questions 1 and 2 for a linear function?

Common Core Mathematical Practices

As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems. Think back over your work, the ways you thought about the Problems, and how you used Mathematical Practices.

Jayden described his thoughts in the following way:

> We noticed differences between the equations for the growth patterns. In Problem 2.1, the water plant on Ghost Lake equation had an additional factor before the base.
>
> Equations in Investigation 1 were of the form $y =$ some number raised to an exponent. Examples are $y = 2^n$ and $y = 3^{n-1}$. These equations did not have a number in front of the bases, 2 and 3.
>
> In Problem 2.1, there is a number in front of the 2^n. The equation is $a = 1{,}000(2^n)$. In this situation, we start tracking growth at $n = 0$ rather than $n = 1$. So, these graphs have a meaningful y-intercept.
>
> **Common Core Standards for Mathematical Practice**
>
> **MP8** Look for and express regularity in repeated reasoning

- What other Mathematical Practices can you identify in Jayden's reasoning?
- Describe a Mathematical Practice that you and your classmates used to solve a different Problem in this Investigation.

Investigation 3

Growth Factors and Growth Rates

In Investigation 2, you studied exponential growth of plants, mold, and a snake population. You used a whole-number growth factor and the starting value to write an equation and make predictions. In this Investigation, you will study exponential growth with fractional growth factors.

3.1 Reproducing Rabbits
Fractional Growth Patterns

In 1859, English settlers introduced a small number of rabbits to Australia. The rabbits had no natural predators in Australia, so they reproduced rapidly and ate grasses intended for sheep and cattle.

Did You Know?

In the mid-1990s, there were more than 300 million rabbits in Australia. The damage they caused cost Australian agriculture $600 million per year. In 1995, a deadly rabbit disease was deliberately spread, reducing the rabbit population by about half. However, because rabbits are developing immunity to the disease, the effects of this measure may not last.

Common Core State Standards

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Also 8.F.A.1, 8.F.A.3, 8.F.B.5, A-SSE.A.1a, A-SSE.A.1b, A-CED.A.2, F-IF.B.4, F-IF.B.6, F-IF.C.8b, F-BF.A.1a, F-LE.A.1a, F-LE.A.1c, F-LE.A.2, F-LE.B.5

Problem 3.1

Suppose biologists had counted the rabbits in Australia in the years after English settlers introduced them. The biologists might have collected data like those shown in the table.

Growth of Rabbit Population

Time (yr)	Population
0	100
1	180
2	325
3	583
4	1,050

A The table shows the rabbit population growing exponentially.

1. What is the growth factor? Explain how you found your answer.
2. Assume this growth pattern continued. Write an equation for the rabbit population p for any year n after the biologists first counted the rabbits. Explain what the numbers in your equation represent.
3. How many rabbits will there be after 10 years? How many will there be after 25 years? After 50 years?
4. In how many years will the rabbit population exceed one million?

B Suppose that, during a different time period, biologists could predict the rabbit population using the equation $p = 15(1.2)^n$, where p is the population in millions, and n is the number of years.

1. What is the growth factor?
2. What was the initial population?
3. In how many years will the initial population double?
4. What will the population be after 3 years? After how many more years will the population at 3 years double?
5. What will the population be after 10 years? After how many more years will the population at 10 years double?
6. How do the doubling times for parts (3)–(5) compare? Do you think the doubling time will be the same for this relationship no matter where you start the count? Explain your reasoning.

ACE Homework starts on page 48.

3.2 Investing for the Future

Growth Rates

The yearly growth factor for one of the rabbit populations in Problem 3.1 is about 1.8. Suppose the population data fit the equation $p = 100(1.8)^n$ exactly. Then its table would look like the one below.

Growth of Rabbit Population

n	p
0	100
1	$100 \times 1.8 =$ 180
2	$180 \times 1.8 =$ 324
3	$324 \times 1.8 =$ 583.2
4	$583.2 \times 1.8 =$ 1,049.76

- Does it make sense to have a fractional part of a rabbit?
- What does this say about the reasonableness of the equation?

The *growth factor* 1.8 is the ratio of the population for a year divided by the population for the previous year. That is, the population for year $n + 1$ is 1.8 times the population for year n.

You can think of the growth factor in terms of a percent change. To find the percent change, compare the difference in population for two consecutive years, n and $n + 1$, with the population of year, n.

- From year 0 to year 1, the percent change is $\frac{180 - 100}{100} = \frac{80}{100} = 80\%$. The population of 100 rabbits in year 0 increased by 80%, resulting in $100 \times 80\% = 80$ additional rabbits.
- From year 1 to year 2, the percent change is $\frac{324 - 180}{180} = \frac{144}{180} = 80\%$. The population of 180 rabbits in year 1 increased by 80%, resulting in $180 \times 80\% = 144$ additional rabbits.

The percent increase is called the **growth rate.** In some growth situations, the growth rate is given instead of the growth factor. For example, changes in the value of investments are often expressed as percents.

- How are the growth rate 80% and the growth factor 1.8 related to each other?

Problem 3.2

When Sam was in seventh grade, his aunt gave him a stamp worth $2,500. Sam considered selling the stamp, but his aunt told him that, if he saved it, it would increase in value.

A Sam saved the stamp, and its value increased by 6% each year for several years in a row.

1. Make a table showing the value of the stamp each year for the five years after Sam's aunt gave it to him.
2. Look at the pattern of growth from one year to the next. Is the value growing exponentially? Explain.
3. Write an equation for the value v of Sam's stamp after n years.
4. How many years will it take to double the value?

B Suppose the value of the stamp increased 4% each year instead of 6%.

1. Make a table showing the value of the stamp each year for the five years after Sam's aunt gave it to him.
2. What is the growth factor from one year to the next?
3. Write an equation that represents the value of the stamp for any year.
4. How many years will it take to double the value?
5. How does the change in percent affect the graphs of the equations?

C 1. Find the growth factor associated with each growth rate.

a. 0% **b.** 15% **c.** 30%

d. 75% **e.** 100% **f.** 150%

2. How you can find the growth factor if you know the growth rate?

D 1. Find the growth rate associated with each growth factor.

a. 1.5 **b.** 1.25 **c.** 1.1 **d.** 1

2. How can you find the growth rate if you know the growth factor?

ACE Homework starts on page 48.

Did You Know?

Some investors use a rule of thumb called the "Rule of 72" to approximate how long it will take the value of an investment to double. To use this rule, simply divide 72 by the annual interest rate.

For example, an investment at an 8% interest rate will take approximately 72 ÷ 8, or 9, years to double. At a 10% interest rate, the value of an investment will double approximately every 7.2 years. This rule doesn't give you exact doubling times, only approximations.

- Do the doubling times you found in Problem 3.2 fit this rule?

3.3 Making a Difference

Connecting Growth Rate and Growth Factor

In Problem 3.2, the value of Sam's stamp increased by the same percent each year. However, each year, this percent was applied to the previous year's value. So, for example, the increase from year 1 to year 2 is 6% of $2,650, not 6% of the original $2,500. This type of change is called **compound growth.**

In this Problem, you will continue to explore compound growth. You will consider the effects of both the initial value and the growth factor on the value of an investment.

Problem 3.3

Mrs. Ramos started college funds for her two granddaughters. She gave $1,250 to Cassie and $2,500 to Kaylee. Mrs. Ramos invested each fund in a 10-year bond that pays 4% interest a year.

A 1. Write an equation to show the relationship between the number of years and the amount of money in each fund.

2. Make a table to show the amount in each fund for 0 to 10 years.

3. Compare the graphs of each equation you wrote in part (1).

4. a. How does the initial value of the fund affect the yearly value increases?

b. How does the initial value affect the growth factor?

c. How does the initial value affect the final value?

B A year later, Mrs. Ramos started a fund for Cassie's cousin, Matt. Cassie made this calculation to predict the value of Matt's fund several years from now:

$$\text{Value} = \$2{,}000 \times 1.05 \times 1.05 \times 1.05 \times 1.05$$

1. What initial value, growth rate, growth factor, and number of years is Cassie assuming?

2. If the value continues to increase at this rate, how much would the fund be worth in one more year?

C Cassie's and Kaylee's other grandmother offers them a choice between college fund options.

Option 1		Option 2
$1,000 at 3% interest per year	OR	$800 at 6% per year

Which is the better option? Explain your reasoning.

ACE Homework starts on page 48.

Applications

1. In parts of the United States, wolves are being reintroduced to wilderness areas where they had become extinct. Suppose 20 wolves are released in northern Michigan, and the yearly growth factor for this population is expected to be 1.2.

   a. Make a table showing the projected number of wolves at the end of each of the first 6 years.

   b. Write an equation that models the growth of the wolf population.

   c. How long will it take for the new wolf population to exceed 100?

2. This table shows the growth of the elk population in a state forest.

   a. The table shows that the elk population is growing exponentially. What is the growth factor? Explain how you found it.

Growth of Elk Population

Time (yr)	Population
0	30
1	57
2	108
3	206
4	391
5	743

   b. Suppose this growth pattern continues. How many elk will there be after 10 years? How many elk will there be after 15 years?

   c. Write an equation you could use to predict the elk population p for any year n after the elk were first counted.

   d. In how many years will the population exceed one million?

3. Suppose there are 100 trout in a lake and the yearly growth factor for the population is 1.5. How long will it take for the number of trout to double?

4. Suppose there are 500,000 squirrels in a forest and the growth factor for the population is 1.6 per year. Write an equation you could use to find the squirrel population p in n years.

5. **Multiple Choice** The equation $p = 200(1.1)^t$ models the exponential growth of a population. The variable p is the population in millions and t is the time in years. How long will it take this population to double?

   A. 4 to 5 years **B.** 5 to 6 years **C.** 6 to 7 years **D.** 7 to 8 years

In Exercises 6 and 7, the equation models the exponential growth of a population, where p is the population in millions and t is the time in years. Tell how much time it would take the population to double.

6. $p = 135(1.7)^t$

7. $p = 1{,}000(1.2)^t$

8. **a.** Fill in the table for each equation.

$y = 50(2.2)^x$

x	0	1	2	3	4	5
y	■	■	■	■	■	■

$y = 350(1.7)^x$

x	0	1	2	3	4	5
y	■	■	■	■	■	■

   b. What is the growth factor for each equation?

   c. Predict whether the graphs of these equations will ever cross.

   d. Estimate any points at which you think the graphs will cross.

9. Maya's grandfather opened a savings account for her when she was born. He opened the account with $100 and did not add or take out any money after that. The money in the account grows at a rate of 4% per year.

   a. Make a table to show the amount in the account from the time Maya was born until she turned 10.

   b. What is the growth factor for the account?

   c. Write an equation for the value of the account after any number of years.

Find the growth rate associated with the given growth factor.

10. 1.4 **11.** 1.9 **12.** 1.75

Find the growth factor associated with the given growth rate.

13. 45% **14.** 90% **15.** 31%

16. Suppose the price of an item increases by 25% per year. What is the growth factor for the price from year to year?

17. Currently, 1,000 students attend Greenville Middle School. The school can accommodate 1,300 students. The school board estimates that the student population will grow by 5% per year for the next several years.

a. When will the population outgrow the present building?

b. Suppose the school limits its growth to 50 students per year. How many years will it take for the population to outgrow the school?

18. Suppose that, for several years, the number of radios sold in the United States increased by 3% each year.

a. Suppose one million radios sold in the first year of this time period. About how many radios sold in each of the next 6 years?

b. Suppose only 100,000 radios sold in the first year. About how many radios sold in each of the next 6 years?

19. Suppose a movie ticket costs about $7, and inflation causes ticket prices to increase by 4.5% a year for the next several years.

a. How much will a ticket cost 5 years from now?

b. How much will a ticket cost 10 years from now? 30 years from now?

c. How many years will it take for the cost of a ticket to exceed $26?

20. Find the growth rate (percent growth) for an exponential function represented by the equation $y = 30(2)^x$.

21. Multiple Choice Ms. Diaz wants to invest $500 in a savings bond. At which bank would her investment grow the most over 8 years?

A. Bank 1: 7% annual interest for 8 years

B. Bank 2: 2% annual interest for the first 4 years and 12% annual interest for the next four years

C. Bank 3: 12% annual interest for the first 4 years and 2% annual interest for the next four years

D. All three result in the same growth.

22. Oscar made the following calculation to predict the value of his baseball card collection several years from now:

$$\text{Value} = \$130 \times 1.07 \times 1.07 \times 1.07 \times 1.07 \times 1.07$$

a. What initial value, growth rate, growth factor, and number of years is Oscar assuming?

b. If the value continues to increase at this rate, how much would the collection be worth in three more years?

23. Carlos, Latanya, and Mila work in a biology laboratory. Each of them is responsible for a population of mice.

The growth factor for Carlos's population of mice is $\frac{3}{8}$.	The growth factor for Latanya's population of mice is 3.	The growth factor for Mila's population of mice is 125%.

a. Whose mice are reproducing fastest?

b. Whose mice are reproducing slowest?

Connections

Calculate each percent.

24. 120% of $3,000

25. 150% of $200

26. 133% of $2,500

For Exercises 27–30, tell whether the pattern represents exponential growth. Explain your reasoning. If the pattern is exponential, give the growth factor.

27. 1 1.1 1.21 1.331 1.4641 1.61051 1.771561

28. 3 5 $8\frac{1}{3}$ $13\frac{8}{9}$ $23\frac{4}{27}$

29. 3 $4\frac{2}{3}$ $6\frac{1}{3}$ 8 $9\frac{2}{3}$ $11\frac{1}{3}$

30. 2 6.4 20.5 66 210

31. A worker currently receives a yearly salary of $20,000.

a. Find the dollar values of a 3%, 4%, and 5% raise for this worker.

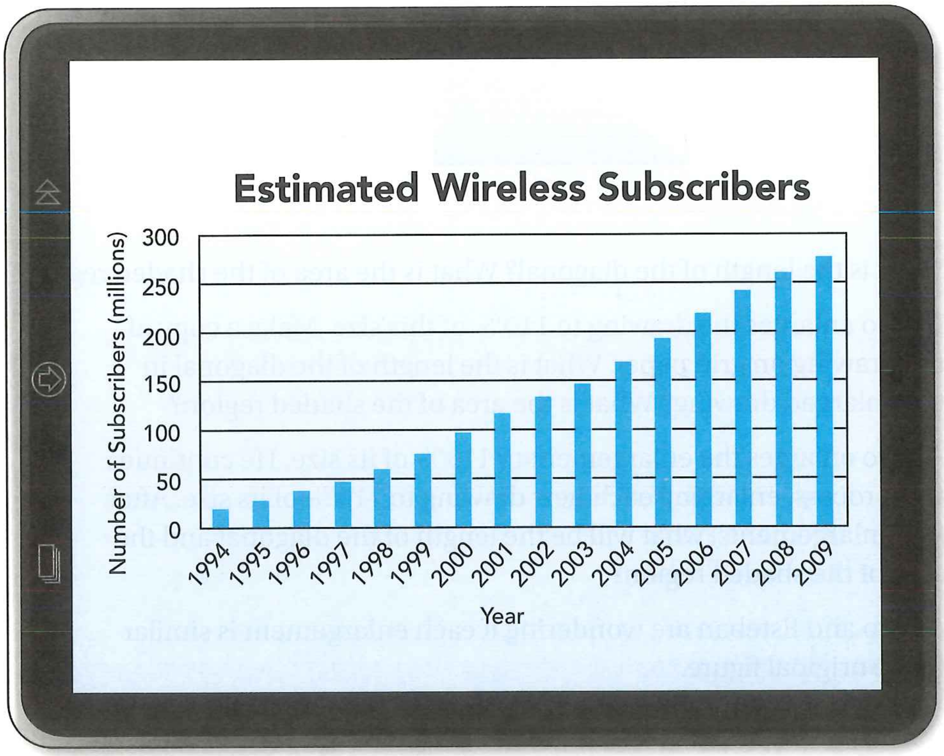
b. Find the worker's new annual salary for each raise in part (a).

c. Joanne says that she can find the new salary with a 3% raise in two ways:

Method 1		Method 2
Add $20,000 to (3% of $20,000).	OR	Find 103% of $20,000.

Explain why these two methods give the same result.

32. The graph shows the growth in the number of wireless subscribers in the United States from 1994 to 2009.



a. What do the bars in the graph represent?

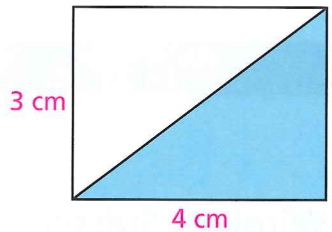
b. What does the implied curve represent?

c. Describe the pattern of change in the total number of subscribers from 1994 to 2009. Could the pattern be modeled by an exponential function or a linear function? Explain.

d. The number of subscribers in 2010 was 300,520,098. In 2011, the number was 322,857,207. Do these numbers fit the pattern you described in part (c)? Explain.

e. If the U.S. population in 2010 was approximately 308 million, what might explain the number of subscriptions from 2011?

33. Refer to the drawing below.



a. What is the length of the diagonal? What is the area of the shaded region?

b. Arturo enlarges the drawing to 110% of this size. Make a copy of the drawing on grid paper. What is the length of the diagonal in the enlarged drawing? What is the area of the shaded region?

c. Arturo enlarges the enlargement to 110% of its size. He continues this process, enlarging each new drawing to 110% of its size. After five enlargements, what will be the length of the diagonal and the area of the shaded region?

d. Arturo and Esteban are wondering if each enlargement is similar to the original figure.

Auturo's Conjecture

All the rectangles are similar because the ratio new width : new length is always 3 : 4. This ratio is the same as the ratio of the width to the length in the original figure.

Esteban's Conjecture

In part (a), the ratio diagonal length : area was different from the same ratio in part (b). Therefore, the figures are not similar.

Which conjecture do you think is correct? Explain. Why is the other conjecture incorrect?

34. Kwan cuts lawns every summer to make money. One customer offers to give her a 3% raise next summer and a 4% raise the summer after that.

Kwan says she would prefer to get a 4% raise next summer and a 3% raise the summer after that. She claims she will earn more money this way. Is she correct? Explain.

35. After graduating from high school, Kim accepts a job with a package delivery service, earning \$9 per hour.

a. How much will Kim earn in a year if she works 40 hours per week for 50 weeks and gets 2 weeks of paid vacation time?

b. Write an equation showing the relationship between the number of weeks Kim works w and the amount she earns a.

c. Kim writes the following equation: $9{,}000 = 360w$. What question is she trying to answer? What is the answer to that question?

d. Suppose Kim works for the company for 10 years, receiving a 3% raise each year. Make a table showing how her annual income grows over this time period.

e. When Kim was hired, her manager told her that instead of a 3% annual raise, she could choose to receive a \$600 raise each year. How do the two raise plans compare over a 10-year period? Which plan do you think is better? Explain your answer.

36. Which represents faster growth, a growth factor of 2.5 or a growth rate of 25%?

37. Order these scale factors from least to greatest.

130% $\quad \frac{3}{2} \quad$ 2 $\quad$ 1.475

38. Christopher made a drawing that measures $8\frac{1}{2}$ by 11 inches. He needs to reduce it so it will fit into a space that measures $7\frac{1}{2}$ by 10 inches. What scale factor should he use to get a similar drawing that is small enough to fit? (Do not worry about getting it to fit perfectly.)

39. a. Match each growth rate from List 1 with the equivalent growth factor in List 2 if possible.

List 1

20%, 120%, 50%, 200%, 400%, 2%

List 2

1.5, 5, 1.2, 2.2, 4, 2, 1.02

b. Order the growth rates from List 1 from least to greatest.

c. Order the growth factors from List 2 from least to greatest.

Extensions

40. In Russia, shortly after the breakup of the Soviet Union, the yearly growth factor for inflation was 26. What growth rate (percent increase) is associated with this growth factor? We call this percent increase the *inflation rate*.

41. In 2000, the population of the United States was about 282 million and was growing exponentially at a rate of about 1% per year.

a. At this growth rate, what will the population of the United States be in the year 2020?

b. At this rate, how long will it take the population to double?

c. The population in 2010 was about 308 million. How accurate was the growth rate?

42. Use the table to answer parts (a)–(d).

a. One model of world population growth assumes the population grows exponentially. Based on the data in this table, what would be a reasonable growth factor for this model?

b. Use your growth factor from part (a) to write an equation for the growth of the population at 5-year intervals beginning in 1955.

c. Use your equation from part (b) to predict the year in which the population was double the 1955 population.

d. Use your equation to predict when the population will be double the 2010 population.

World Population Growth

Year	Population (billions)
1955	2.76
1960	3.02
1965	3.33
1970	3.69
1975	4.07
1980	4.43
1985	4.83
1990	5.26
1995	5.67
2000	6.07
2005	6.46
2010	6.84

For Exercises 43–45, write an equation that represents the exponential function in each situation.

43. A population is initially 300. After 1 year, the population is 361.

44. A population has a yearly growth factor of 1.2. After 3 years, the population is 1,000.

45. The growth rate for an investment is 3% per year. After 2 years, the value of the investment is $2,560.

46. Suppose your calculator did not have an exponent key. You could find 1.5^{12} by entering:

$1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5 \times 1.5$

a. How could you evaluate 1.5^{12} with fewer keystrokes?

b. What is the fewest times you could press [×] to evaluate 1.5^{12}?

47. Mr. Watson sold his boat for $10,000. He wants to invest the money.

Growth of $10,000 Investment at 4% Interest Compounded Quarterly

Time (mo)	Money in Account
0	$10,000
3	$10,100
6	$10,201
9	$10,303.01

a. How much money will Mr. Watson have after 1 year if he invests the $10,000 in an account that pays 4% interest per year?

b. Mr. Watson sees an advertisement for another type of savings account:

"4% interest per year compounded quarterly."

He asks the bank teller what "compounded quarterly" means. She explains that instead of giving him 4% of $10,000 at the end of one year, the bank will give him 1% at the end of each 3-month period (each quarter of a year).

If Mr. Watson invests his money at this bank, how much will be in his account at the end of one year?

c. Mr. Watson sees an advertisement for a different bank that offers 4% interest per year *compounded monthly*. (This means he will get $\frac{1}{12}$ of 4% interest every month.) How much money will he have at the end of the year if he invests his money at this bank?

d. Which account would have the most money at the end of one year? Explain.

Mathematical Reflections 3

In this Investigation, when you were given tables and descriptions of relationships, you explored exponential functions in which the growth factor was not a whole number. In some of these situations, the growth was described by giving the percent growth, or growth rate.

Think about these questions. Discuss your ideas with other students and your teacher. Then write a summary of your findings in your notebook.

1. Suppose you know the initial value for a population and the yearly growth rate.
   a. **How** can you determine the population several years from now?
   b. **How** is a growth rate related to the growth factor for the population?
   c. **How** can you use this information to write an equation that models the situation?
2. Suppose you know the initial value for a population and the yearly growth factor.
   a. **How** can you determine the population several years from now?
   b. **How** can you determine the yearly growth rate?
3. Suppose you know the equation that represents the exponential function relating the population p and the number of years n. **How** can you determine the doubling time for the population?

Common Core Mathematical Practices

As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems. Think back over your work, the ways you thought about the Problems, and how you used Mathematical Practices.

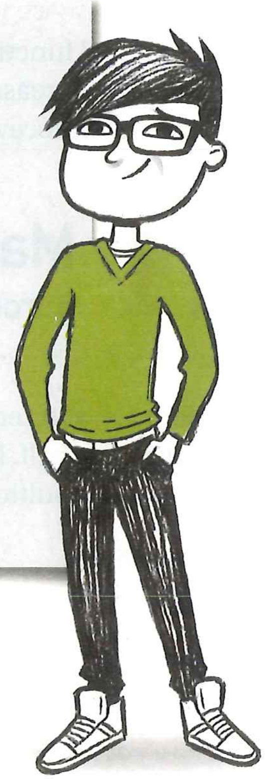
Nick described his thoughts in the following way:

We thought that the growth of the rabbit population in Problem 3.1 represented an exponential function.

We compared the population from one year with the next year's population. We found that the ratio was not exactly, but was very close to, 1.8.

This data is experimental. Factors such as food availability and weather conditions could affect the growth of the population from year to year. So, we used 1.8 as an approximation of the growth rate.

Common Core Standards for Mathematical Practice
MP4 Model with mathematics



- What other Mathematical Practices can you identify in Nick's reasoning?
- Describe a Mathematical Practice that you and your classmates used to solve a different Problem in this Investigation.

Investigation 4

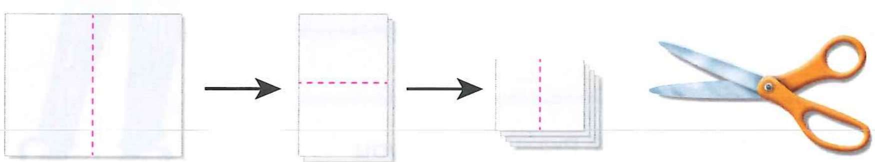
Exponential Decay

The exponential functions you have studied so far have all involved quantities that increase. In this Investigation, you will explore quantities that decrease, or decay, exponentially as time passes.

4.1 Making Smaller Ballots

Introducing Exponential Decay

In Problem 1.1, you read about the ballots Chen is making. Chen cuts a sheet of paper in half. He stacks the two pieces and cuts them in half. Chen then stacks the resulting four pieces and cuts them in half, and so on.



Common Core State Standards

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

8.F.A.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear.

8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph . . . Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Also 8.F.A.1, A-SSE.A.1a, A-SSE.A.1b, A-CED.A.2, A-REI.D.10, F-IF.B.4, F-IF.B.6, F-IF.C.7e, F-BF.A.1a, F-LE.A.1a, F-LE.A.1c, F-LE.A.2, F-LE.B.5

You investigated the pattern in the number of ballots each cut made. In this Problem, you will look at the pattern in the areas of the ballots.

Problem 4.1

A The paper Chen starts with has an area of 64 square inches. Copy and complete the table to show the area of a ballot after each of the first 10 cuts.

Areas of Ballots

Number of Cuts	Area (in.^{2})
0	64
1	32
2	16
3	■
4	■
5	■
6	■
7	■
8	■
9	■
10	■

B How does the area of a ballot change with each cut?

C Write an equation for the area A of a ballot after any cut n.

D Make a graph of the data.

E 1. How is the pattern of change in the area different from the exponential growth patterns you studied? How is it similar?

2. How is the pattern of change in the area different from linear patterns you studied? How is it similar?

ACE Homework starts on page 66.

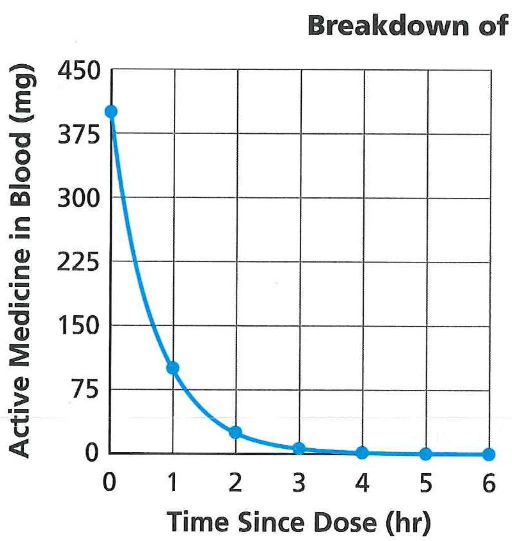
4.2 Fighting Fleas
Representing Exponential Decay

Exponential patterns like the one in Problem 4.1, in which a quantity decreases at each stage by a constant factor, show **exponential decay.** The factor the quantity is multiplied by at each stage is called the **decay factor.** A decay factor is always greater than 0 and less than 1. In Problem 4.1, the decay factor is $\frac{1}{2}$.

- Are exponential decay patterns also exponential functions? Explain.

After an animal receives flea medicine, the medicine breaks down in the animal's bloodstream. With each hour, there is less medicine in the blood.

A dog receives a 400-milligram dose of flea medicine. The table and graph show the amount of medicine in the dog's bloodstream each hour for 6 hours after the dose.



Time Since Dose (hr)	Active Medicine in Blood (mg)
0	400
1	100
2	25
3	6.25
4	1.5625
5	0.3907
6	0.0977

Problem 4.2

A Study the pattern of change in the graph and the table.

1. How does the amount of active medicine in the dog's blood change from one hour to the next?
2. Write an equation to model the relationship between the number of hours h since the dose is given and the milligrams of active medicine m.
3. How is the graph for this Problem similar to the graph you made in Problem 4.1? How is it different?
4. Does the relationship displayed in the table and graph represent an exponential function? Explain.

B 1. A different flea medicine breaks down at a rate of 20% per hour. This means that as each hour passes, 20% of the active medicine is used. This is the **rate of decay** of the medicine. The initial dose is 60 milligrams. Extend and complete this table to show the amount of active medicine in an animal's blood at the end of each hour for 6 hours.

Breakdown of Medicine

Time Since Dose (hr)	Active Medicine in Blood (mg)
0	60
1	■
2	■
⋮	⋮
6	■

2. Make a graph using the table you completed in part (1).
3. For the medicine in part (1), Janelle wrote the equation $m = 60(0.8)^h$ to show the amount of active medicine m after h hours. Compare the quantities of active medicine in your table to the quantities given by Janelle's equation. Explain any similarities or differences.
4. Dwayne was confused by the terms *decay rate* (or *rate of decay*) and *decay factor*. He said:

> Because the rate of decay is 20%, the decay factor should be 0.2, and the equation should be $m = 60(0.2)^h$.

Do you agree with him? Explain.

5. Steven recalled that when the growth rate is 80%, the growth factor is 1.8 or 180%. How is the relationship between growth rate and growth factor similar to the relationship between decay rate and decay factor?

ACE Homework starts on page 66.

4.3 Cooling Water
Modeling Exponential Decay

Sometimes a cup of hot cocoa or tea is too hot to drink at first. So you must wait for it to cool.

- What pattern of change would you expect to find in the temperature of a hot drink as time passes?
- What shape would you expect for a graph of data (*time, drink temperature*)?

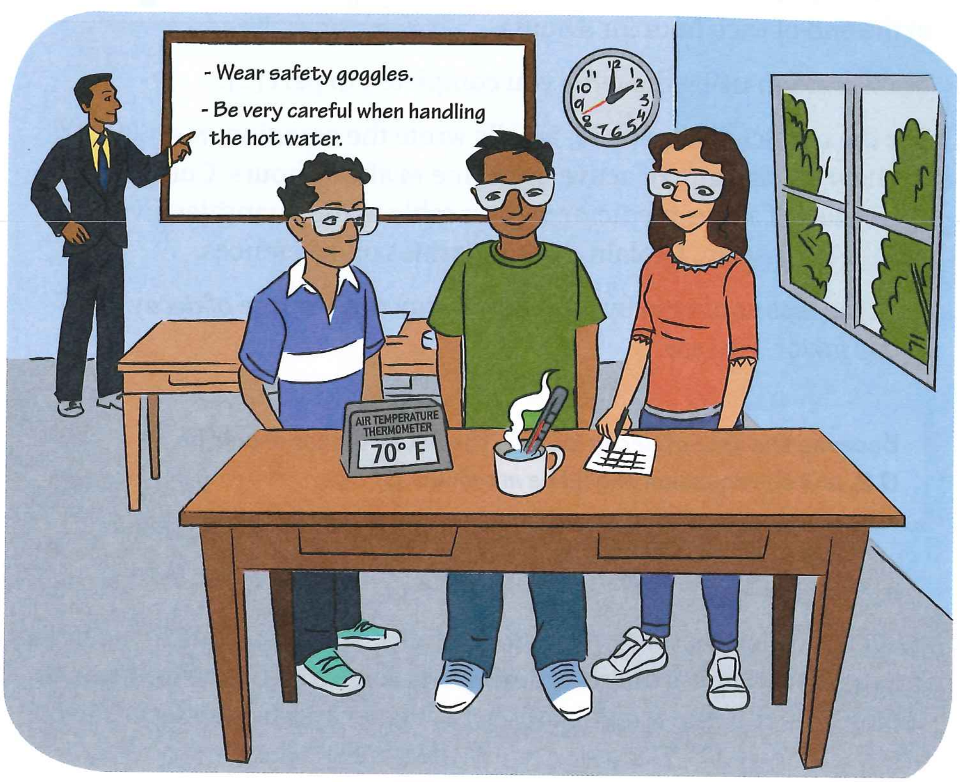
This experiment will help you explore these questions.

Equipment

- very hot water
- two thermometers
- a cup or mug for hot drinks
- a watch or clock

Directions

- Record air temperature.
- Fill the cup/mug with hot water.
- In a table, record the water temperature and room temperature at 5-minute intervals throughout your class period.



Problem 4.3

A **1.** Complete the table with data from your experiment.

Hot Water Cooling

Time (min)	Water Temperature	Room Temperature
0	■	■
5	■	■
10	■	■
■	■	■

Make a graph of your (*time, water temperature*) data.

2. Describe the pattern of change in the data. When did the water temperature change most rapidly? When did it change most slowly?

3. Is the relationship between time and water temperature an exponential decay relationship? Explain.

B **1.** Add a column to your table. In this column, record the difference between the water temperature and the air temperature for each time value.

2. Make a graph of the (*time, temperature difference*) data. Compare this graph with the graph you made in Question A.

3. Describe the pattern of change in the data. When did the temperature difference change most rapidly? Most slowly?

4. Estimate the decay factor for the relationship between temperature difference and time in this experiment.

5. Write an equation for the (*time, temperature difference*) data. Your equation should allow you to predict the temperature difference at the end of any 5-minute interval.

C **1.** What do you think the graph of the (*time, temperature difference*) data would look like if you had continued the experiment for several more hours?

2. What factors might affect the rate at which a cup of hot liquid cools?

3. What factors might introduce errors in the data you collect?

D Compare the graphs in Questions A and B with the graphs in Problems 4.1 and 4.2. What similarities and differences do you observe?

ACE Homework starts on page 66.

Applications

1. Chen, from Problem 4.1, finds that his ballots are very small after only a few cuts. He decides to start with a larger sheet of paper. The new paper has an area of 324 in.^{2}. Copy and complete this table to show the area of each ballot after each of the first 10 cuts.

Areas of Ballots

Number of Cuts	Area (in.^{2})
0	324
1	162
2	81
3	■
4	■
5	■
6	■
7	■
8	■
9	■
10	■

a. Write an equation for the area A of a ballot after any cut n.

b. With the smaller sheet of paper, the area of a ballot is 1 in.^{2} after 6 cuts. Start with the larger sheet. How many cuts does it take to get ballots this small?

c. Chen wants to be able to make 12 cuts before getting ballots with an area of 1 in.^{2}. How large does his starting piece of paper need to be?

2. During the exploration of Problem 4.1, several groups of students in Mrs. Dole's class made a conjecture. They conjectured that the relationship between the number of cuts and the area of the ballot was an *inverse variation* relationship.

The class came up with two different arguments for why the relationship was not an inverse variation.

Argument 1

An inverse variation situation has a "factor-pair" relationship. Choose some constant number k. The two factors multiply to equal k, such as $yx = k$. For example, if the area of rectangle with length, l, and width, w, is 24,000 square feet, then $24{,}000 = lw$. This is an inverse variation.

In an exponential relationship, the values of the two variables x and y do not have this "factor-pair" relationship. For example, in Problem 4.1, the equation is $A = 64\left(\frac{1}{2}\right)^n$, but A and n do not multiply to get a constant number.

Argument 2

Any inverse variation will never have a y-intercept and this relationship does. Therefore, this relationship is not an inverse variation.

Which argument is correct? Explain why the students might have made this conjecture.

3. Latisha has a 24-inch string of licorice (LIK uh rish) to share with her friends. As each friend asks her for a piece, Latisha gives him or her half of what she has left. She doesn't eat any of the licorice herself.

   a. Make a table showing the length of licorice Latisha has left each time she gives a piece away.

   b. Make a graph of the data from part (a).

   c. Suppose that, instead of half the licorice that is left each time, Latisha gives each friend 4 inches of licorice. Make a table and a graph for this situation.

   d. Compare the tables and the graphs for the two situations. Explain the similarities and the differences.

4. Penicillin decays exponentially in the human body. Suppose you receive a 300-milligram dose of penicillin to combat strep throat. About 180 milligrams will remain active in your blood after 1 day.

   a. Assume the amount of penicillin active in your blood decreases exponentially. Make a table showing the amount of active penicillin in your blood for 7 days after a 300-milligram dose.

   b. Write an equation for the relationship between the number of days d since you took the penicillin and the amount of the medicine m remaining active in your blood.

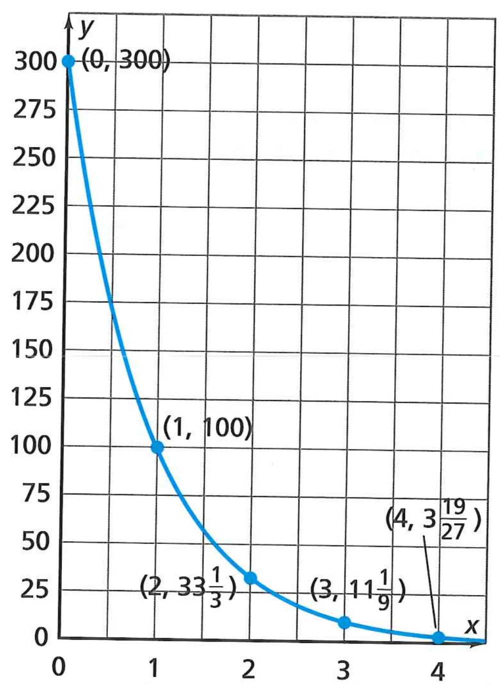
   c. What is the equation for a 400-milligram dose?

For Exercises 5 and 6, tell whether the equation represents exponential decay or exponential growth. Explain your reasoning.

5. $y = 0.8(2.1)^x$

6. $y = 20(0.5)^x$

7. The graph below shows an exponential decay relationship.



   a. Find the decay factor and the y-intercept.

   b. What is the equation for the graph?

For Exercises 8 and 9, use the table of values to determine the exponential decay equation. Then, find the decay factor and the decay rate.

8.

x	y
0	24
1	6
2	1.5
3	0.375
4	0.09375

9.

x	y
0	128
1	96
2	72
3	54

For Exercises 10–13, use Lara's conjecture below. Explain how you found your answer.

> **Lara's Conjecture**
>
> If you know the y-intercept and another point on the graph of an exponential function, then you can find all the other points.

10. The exponential decay graph has y-intercept $= 90$, and it passes through $(2, 10)$. When $x = 1$, what is y?

11. The exponential decay graph has y-intercept $= 40$, and it passes through $(2, 10)$. When $x = 4$, what is y?

12. The exponential decay graph has y-intercept $= 75$, and it passes through $(2, 3)$. When $x = -2$, what is y?

13. The exponential decay graph has y-intercept $= 64$, and it passes through $(3, 0.064)$. When $x = 2$, what is y?

14. Karen shops at Aquino's Groceries. Her bill came to $50 before tax. She used two of the coupons shown below.

Karen was expecting to save 10%, which is $5. The cashier rang up the two coupons. Karen was surprised when the total price rang up as $45.13 before tax. She was not sure why there was an extra $0.13 charge.

a. What would explain why the coupons did not take off 10% the way Karen expected?

b. Write an equation to represent the total amount Karen would spend based on the number of coupons she would use.

c. Karen had originally thought that if she used 10 coupons on her next trip to Aquino's Groceries she would save 50%. Her bill is still $50. How much would Karen actually spend?

d. How many coupons would you estimate it would take for Karen to get the $50 of groceries for free?

15. Hot coffee is poured into a cup and allowed to cool. The difference between coffee temperature and room temperature is recorded every minute for 10 minutes.

Cooling Coffee

Time (min)	0	1	2	3	4	5	6	7	8	9	10
Temperature Difference (°C)	80	72	65	58	52	47	43	38	34	31	28

a. Plot the data (*time, temperature difference*). Explain what the patterns in the table and the graph tell you about the rate at which the coffee cools.

b. Approximate the decay factor for this relationship.

c. Write an equation for the relationship between time and temperature difference.

d. About how long will it take the coffee to cool to room temperature? Explain.

16. The pizza in the ad for Mr. Costa's restaurant has a diameter of 5 inches.

a. What are the circumference and area of the pizza in the ad?

b. Mr. Costa reduces his ad to 90% of its original size. He then reduces the reduced ad to 90% of its size. He repeats this process five times. Extend and complete the table to show the diameter, circumference, and area of the pizza after each reduction.

Advertisement Pizza Sizes

Reduction Number	Diameter (in.)	Circumference (in.)	Area ($in.^2$)
0	5	■	■
1	■	■	■

c. Write equations for the diameter, circumference, and area of the pizza after *n* reductions.

d. How would your equations change if Mr. Costa had used a reduction setting of 75%?

e. Express the decay factors from part (d) as fractions.

f. Mr. Costa claims that when he uses the 90% reduction setting on the copier, he is reducing the size of the drawing by 10%. Is Mr. Costa correct? Explain.

17. Answer parts (a) and (b) without using your calculator.

a. Which decay factor represents faster decay, 0.8 or 0.9?

b. Order the following from least to greatest:

0.9^4 0.9^2 90% $\frac{2}{10}$ $\frac{2}{9}$ 0.8^4 0.84

18. Natasha and Michaela are trying to find growth factors for exponential functions. They claim that if the independent variable is increasing by 1, then you divide the two corresponding *y* values to find the growth factor. For example, if (x_1, y_1) and (x_2, y_2) are two consecutive entries in the table, then the growth factor is $y_2 \div y_1$.

a. Is their reasoning correct? Explain.

b. Would this method work to find the growth pattern for a linear function? Explain.

Connections

For Exercises 19–22, write each number in scientific notation.

19. There are about 33,400,000,000,000,000,000,000 molecules in 1 gram of water.

20. There are about 25,000,000,000,000 red blood cells in the human body.

21. Earth is about 93,000,000 miles (150,000,000 km) from the sun.

22. The Milky Way galaxy is approximately 100,000 light years in diameter. It contains about 300,000,000,000 stars.

23. Consider these equations:

$$y = 0.75^x \qquad y = 0.25^x \qquad y = -0.5x + 1$$

a. Sketch graphs of all three equations on one set of coordinate axes.

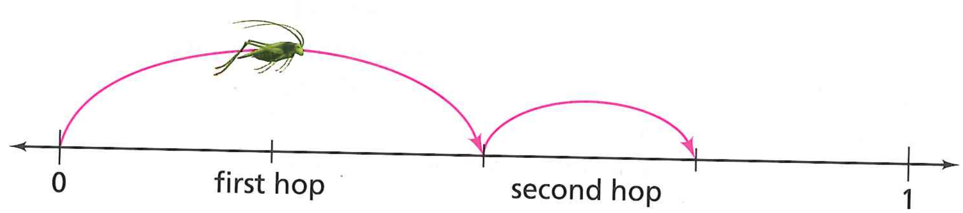
b. What points, if any, do the three graphs have in common?

c. In which graph does y decrease the fastest as x increases?

d. How can you use your graphs to figure out which of the equations is not an example of exponential decay?

e. How can you use the equations to figure out which is not an example of exponential decay?

24. A cricket is on the 0 point of a number line, hopping toward 1. She covers half the distance from her current location to 1 with each hop. So, she will be at $\frac{1}{2}$ after one hop, $\frac{3}{4}$ after two hops, and so on.



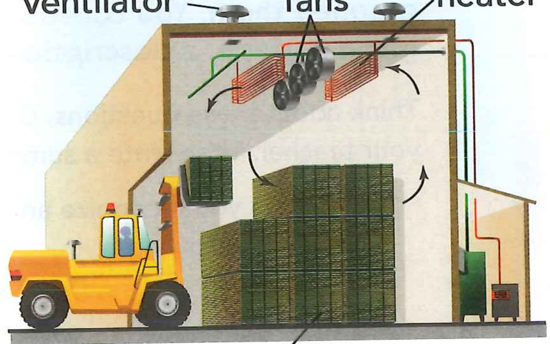
a. Make a table showing the cricket's location for the first 10 hops.

b. Where will the cricket be after n hops?

c. Will the cricket ever get to 1? Explain.

Extensions

25. Freshly cut lumber, known as *green lumber*, contains water. If green lumber is used to build a house, it may crack, shrink, and warp as it dries. To avoid these problems, lumber is dried in a kiln that circulates air to remove moisture from the wood.



Suppose that, in 1 week, a kiln removes $\frac{1}{3}$ of the moisture from a stack of lumber.

a. What fraction of the moisture remains in the lumber after 5 weeks in a kiln?

b. What fraction of the moisture has been removed from the lumber after 5 weeks?

c. Write an equation for the fraction of moisture m remaining in the lumber after w weeks.

d. Write an equation for the fraction of moisture m that has been removed from the lumber after w weeks.

e. Graph your equations from parts (c) and (d) on the same set of axes. Describe how the graphs are related.

f. A different kiln removes $\frac{1}{4}$ of the moisture from a stack of lumber each week. Write equations for the fraction of moisture remaining and the fraction of moisture removed after w weeks.

g. Graph your two equations from part (f) on the same set of axes. Describe how the graphs are related. How do they compare to the graphs from part (e)?

h. Green lumber is about 40% water by weight. The moisture content of lumber used to build houses is typically 10% or less. For each of the two kilns described above, how long should lumber be dried before it is used to build a house?

Mathematical Reflections 4

In this Investigation, you explored situations in which a quantity decayed by a constant percent rate per unit interval and graphed them. You constructed exponential decay functions given a graph, a description, or a table.

Think about these questions. Discuss your ideas with other students and your teacher. Then write a summary of your findings in your notebook.

1. **How** can you recognize an exponential decay pattern from the following?

   a. a table of data

   b. a graph

   c. an equation

2. **How** are exponential growth functions and exponential decay functions similar? **How** are they different?

3. **How** are exponential decay functions and decreasing linear functions similar? **How** are they different?

Common Core Mathematical Practices

As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems. Think back over your work, the ways you thought about the Problems, and how you used Mathematical Practices.

Sophie described her thoughts in the following way:

We collected data to determine the pattern of change in the temperature of the water in a cup in Problem 4.3. We started with boiling water and checked the temperature every 5 minutes. Then, we compared the water temperature to the room temperature and recorded the difference.

We then fit a graph to the data. We used the equation to find an approximate decay factor.

This process was very similar to the one we used in determining bridge strength in the Thinking with Mathematical Models Unit.

Common Core Standards for Mathematical Practice

MP7 Look for and make use of structure

- What other Mathematical Practices can you identify in Sophie's reasoning?
- Describe a Mathematical Practice that you and your classmates used to solve a different Problem in this Investigation.

Investigation 5

Patterns with Exponents

As you explored exponential functions in previous Investigations, you made tables of exponential growth. The table shows some values for $y = 2^n$. The y-values are given in both exponential and standard form.

The table shows interesting patterns. For example, Roxanne noticed that in each of the products below, the sum of the exponents of all the factors is the exponent for the product.

$$2^1 \times 2^2 = 2^3$$

$$2^2 \times 2^3 = 2^5$$

- Is Roxanne correct? Explain why or why not.
- Does Roxanne's pattern hold for any number b?
  For instance, what is the value of the expression $b^2 \times b^3$?

$y = 2^n$

x	y
1	2^1 or 2
2	2^2 or 4
3	2^3 or 8
4	2^4 or 16
5	2^5 or 32
6	2^6 or 64
7	2^7 or 128
8	2^8 or 256

Common Core State Standards

8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions.

8.EE.A.2. Use square root and cube root symbols to represent equations of the form $x^2 = p$ and $x^3 = p$. . . .

8.EE.A.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used.

Also 8.F.A.3, A-SSE.A.1a, A-SSE.A.2, A-SSE.B.3c, N-RN.A.1, N-RN.A.2, F-IF.C.7e, F-IF.C.8b, F-LE.B.5

5.1 Looking for Patterns Among Exponents

The values of a^x for a given number a are called *powers of a*. You just looked at situations involving powers of 2. In Problem 5.1, you will explore patterns with powers of 2 and other numbers.

Problem 5.1

A Copy and complete this table.

x	1^x	2^x	3^x	4^x	5^x	6^x	7^x	8^x	9^x	10^x
1	1	2	■	■	■	■	■	■	■	■
2	1	4	■	■	■	■	■	■	■	■
3	1	8	■	■	■	■	■	■	■	■
4	1	16	■	■	■	■	■	■	■	■
5	1	32	■	■	■	■	■	■	■	■
6	1	64	■	■	■	■	■	■	■	■
7	1	128	■	■	■	■	■	■	■	■
8	1	256	■	■	■	■	■	■	■	■

B 1. Describe any patterns that you see in the rows and columns.

2. Give examples for your patterns.

3. Explain why your patterns are correct.

C Delmar noticed that if you extend the pattern upwards in the table for negative values of x, you get the values shown at the right.

1. Copy the table and extend it to include columns for 3^x through 10^x.

2. Delmar claims that $a^0 = 1$ and that $a^{-1} = \frac{1}{a}$ for any positive number a. Do you agree? Explain.

−2	1	$\frac{1}{4}$
−1	$\frac{1}{1} = 1$	$\frac{1}{2}$
0	1	1
x	1^x	2^x

ACE Homework starts on page 88.

5.2 Rules of Exponents

In Problem 5.1, you explored patterns among the values of a^x for different values of a. For example, Federico noticed that 64 appears three times in the powers table. It is in the column for 2^x, 4^x, and 8^x. He said this means that $2^6 = 4^3 = 8^2$.

- Explain why Federico's conclusion is true.
- Are there other numbers that appear more than once in the table?
- What other patterns do you notice in the table?

In Problem 5.2, you will look at patterns that lead to some important properties of exponents.

Problem 5.2

Use your table from Problem 5.1 to help you answer these questions.

A **1.** Write each of the following in expanded form. Then, write the answer in exponential form with a single base and power.

a. $2^3 \times 2^2$ **b.** $3^4 \times 3^3$ **c.** $6^5 \times 6^5$

2. What do you notice when you multiply two powers with the same base?

3. Finish the following equation to express what you noticed. Explain why it is true.

$$a^m \times a^n = \blacksquare$$

B **1.** Rewrite each multiplication sentence as an equivalent division sentence.

a. $3^3 \times 3^2 = 3^5$ **b.** $4^6 \times 4^5 = 4^{11}$ **c.** $5^8 \times 5^4 = 5^{12}$

2. What do you notice when you divide two powers with the same base? Why do you think this happens?

3. Finish the following equation to express what you noticed. Explain your reasoning. Assume $a \neq 0$.

$$\frac{a^m}{a^n} = \blacksquare$$

Problem 5.2 continued

C 1. Write each of the following in expanded form. Then write the result in exponential form with a single base and power.

a. $2^3 \times 5^3$ b. $5^2 \times 6^2$ c. $10^4 \times 2^4$

2. What do you notice when you multiply two powers with the same exponent but different bases?

3. Finish the following equation to express what you noticed. Explain.

$$a^m \times b^m = \blacksquare$$

D 1. Mary says she can use the fact below to write a power raised to a power with a single base and power.

I know that $(2^3)^2 = (2^3) \cdot (2^3)$.

Use that fact to write each of the following with a single base and power.

a. $(3^2)^2$ b. $(5^3)^3$ c. $(2^2)^4$

2. What do you notice when you raise a power to a power?

3. Finish the following equation to express what you know. Explain.

$$(a^m)^n = \blacksquare$$

E As he worked on Problem 5.1, Question C, Delmar made the following claim.

$a^0 = 1$ and $a^{-1} = \frac{1}{a}$.

Use what you have learned in Questions A–D to show why each of the following is true for any nonzero value of a:

1. $a^0 = 1$
2. $a^{-1} = \frac{1}{a}$
3. Finish the following equation.

$$a^{-m} = \blacksquare$$

ACE Homework starts on page 88.

5.3 Extending the Rules of Exponents

In Problem 5.1 and Problem 5.2 you investigated rules for integral exponents and found the following to be true, where $x \neq 0, y \neq 0$, and m and n are integers.

$$x^m x^n = x^{m+n}$$

$$(x^m)^n = x^{mn}$$

$$(xy)^m = x^m y^m$$

$$\frac{x^m}{x^n} = x^{m-n}$$

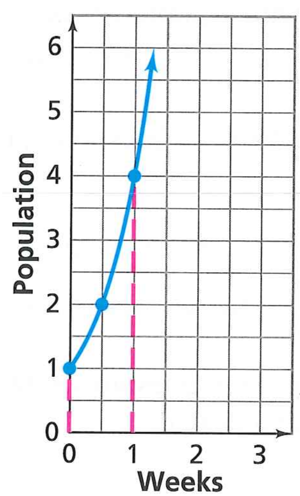
$$x^{-1} = \frac{1}{x}$$

$$x^0 = 1$$

- Do these rules work for rational exponents?

Suppose a certain amoeba population quadruples every week. If you start with 1 amoeba, then the population y grows according to the relationship $y = 4^x$. The graph of this relationship appears below.

Amoeba Population Growth



- Does it make sense to write $y = 4^{\frac{1}{2}}$?
- If so, what point does this correspond to on the graph?

The point $\left(\frac{1}{2}, 2\right)$ is the point on the graph that corresponds to an x-value of $\frac{1}{2}$. This means that $4^{\frac{1}{2}} = 2$. Chaska then reasons as follows.

Since you already know that $\sqrt{4} = 2$, it must be true that $\sqrt{4} = 4^{\frac{1}{2}}$.

- Do the rules for adding exponents apply to the exponent $\frac{1}{2}$?
  For instance, is it true that $4^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}} = 4$?

Chaska then thinks about other roots.

I know that $\sqrt[3]{8} = 2$. So I conclude that the rule for the exponent $\frac{1}{2}$ can be extended to the exponent $\frac{1}{3}$.

- How can you write $\sqrt[3]{8} = 2$ using exponents?
- How can Chaska use the rules for exponents to confirm that $\sqrt[3]{8} \cdot \sqrt[3]{8} \cdot \sqrt[3]{8} = 8$?
- In general, the **nth root** of a number b is denoted by $\sqrt[n]{b}$ or $b^{\frac{1}{n}}$, where n is an integer greater than 1.
- Think about what Chaska found about the exponents $\frac{1}{2}$ and $\frac{1}{3}$.
  Is it true that $\sqrt[n]{x} = x^{\frac{1}{n}}$?

?

- Do the other rules for exponents apply to exponents that are fractions?

In Problem 5.3, you will explore whether the rules for integral exponents apply for rational exponents.

Problem 5.3

A What does $16^{\frac{3}{2}}$ mean?

1. Mari, Latrell, and Jakayla came up with three different ways to think about $16^{\frac{3}{2}}$. Do any of these make sense? Explain.

Mari's Method	Latrell's Method	Jakayla's Method
$16^{\frac{3}{2}} = \left(16^{\frac{1}{2}}\right)^3$	$16^{\frac{3}{2}} = (16^3)^{\frac{1}{2}}$	$16^{\frac{3}{2}} = 16^{1+\frac{1}{2}}$
$= (4)^3$	$= (4096)^{\frac{1}{2}}$ or $\sqrt{4096}$	$= 16^1\left(16^{\frac{1}{2}}\right)$ or $16\sqrt{16}$
$= 64$	$= 64$	$= 16(4)$
		$= 64$

2. Find the value of each of the following. Show your method.

   a. $8^{\frac{5}{3}}$ **b.** $125^{\frac{4}{3}}$

B 1. Use what you learned about roots to compute each of the following.

   a. $4^{\frac{3}{2}} \cdot 4^{\frac{1}{2}}$ **b.** $4^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$ **c.** $\left(2^{\frac{5}{3}}\right)^3$ **d.** $\dfrac{25^{\frac{3}{2}}}{25^{\frac{1}{2}}}$

2. Use the rules for integral exponents to compute the answer in a different way for each of the expressions in part (1). Do you get the same numbers? Explain why or why not.

C Suppose that in Problem 1.2, R is the number of rubas on the nth square of a chessboard. Mari and Latrell came up with the following two equations for R.

Mari	Latrell
$R = \frac{1}{2}(2^n)$	$R = 2^{n-1}$

Which equation is correct? Explain.

Problem 5.3 continued

D Suppose that in Problem 1.4, the number of cuts is n and the area of each piece is A. Jakayla, Mari, and Latrell came up with three ways to express the exponential relationship.

Mari	Latrell	Jakayla
$A = \frac{64}{2^n}$	$A = 64(0.5^n)$	$A = 64(2^{-n})$

Are these all correct? Explain.

E Use the rules of exponents to write an equivalent expression for each of the following.

1. $x^{\frac{1}{2}} \cdot x^{\frac{3}{2}}$
2. $x^{\frac{2}{3}} \div x^{\frac{7}{6}}$
3. $\left(2x^{\frac{1}{2}}\right)^2$
4. $\left(16x^{\frac{4}{3}}\right)^{\frac{1}{2}}$

ACE Homework starts on page 88.

Did You Know?

Having only a single cell, amoebas are among the simplest organisms. Even so, amoebas and humans have common features. Both have DNA and cellular structure. The life cycle of an amoeba is typically a few days.

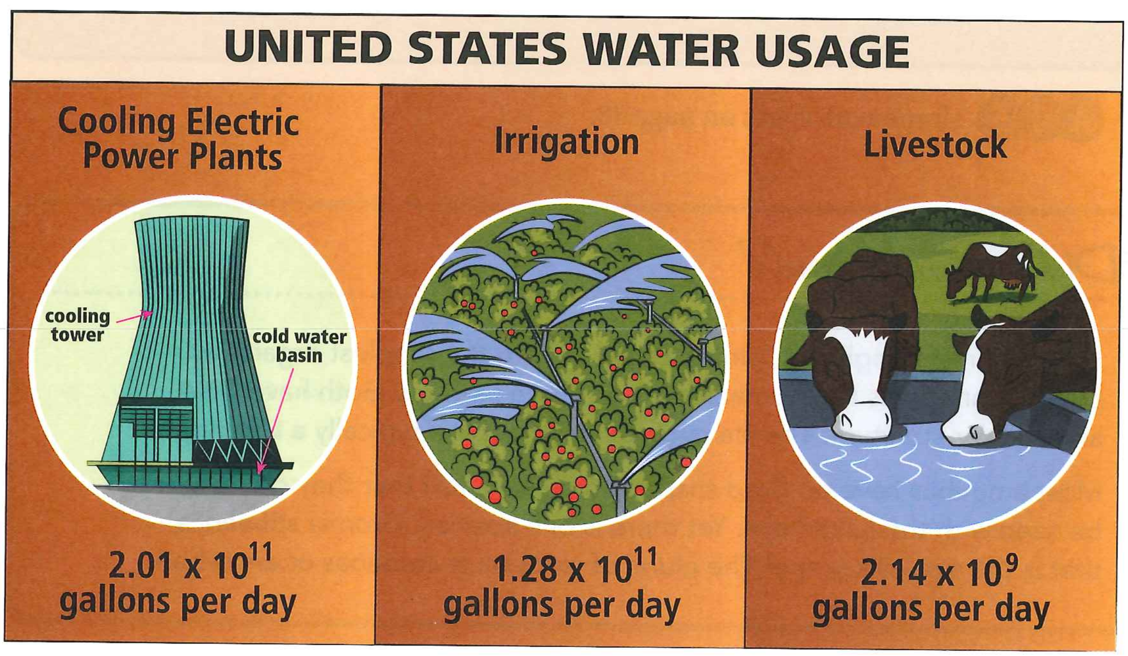
Most amoebas have no fixed shape and are so small that they cannot be seen with the naked eye. Yet there is one species, *Gromia sphaerica*, that is the size of a grape! The plural of amoeba is *amoebas* or *amoebae*.

5.4 Operations with Scientific Notation

Gray water is a term for wastewater that is reused without any treatment. For example, some people use the water that drains from their shower, bathtub, washing machine, or dishwasher to water their gardens. Reusing water in this way helps conserve an important resource.

The United States uses a huge amount of water. To get a sense of the amounts used for various purposes, consider the following figures from a recent year.

- A total of approximately 4.10×10^{11} gallons of water is used each day in the United States.
- Water for cooling electric power plants demands 2.01×10^{11} gallons per day.
- Irrigation uses about 1.28×10^{11} gallons of water per day.
- Livestock consumes about 2.14×10^{9} gallons of water per day.



Gary and Judy are studying water use. They need to figure out how much water is used per person each day. The U.S. population in the same year was approximately 301,000,000. Assume that everyone uses the same amount of water each day.

- How much water is used per person each day?

Problem 5.4

A Gary and Judy are figuring out how much water each person uses per day. They used the following expression to find their answer: $(4.10 \times 10^{11}) \div (301{,}000{,}000)$. Each of them used a different method for carrying out the calculations. Consider their two methods for solving the problem.

Gary's Method

I thought of both numbers in millions. 10^6 is one million.
So $4.10 \times 10^{11} = 410{,}000 \times 10^6$, and $301{,}000{,}000 = 301 \times 10^6$.
Dividing gives me $410{,}000 \div 301 \approx 1{,}360$.
My result is 1,360 gallons per person per day.

Judy's Method

I converted 301,000,000 to 3.01×10^8. Then I rewrote the problem as

$$\frac{4.10 \times 10^{11}}{3.01 \times 10^8} = \frac{4.10}{3.01} \times \frac{10^{11}}{10^8} \approx 1.36 \times 10^3.$$

I know that $10^{11} \div 10^8$ equals 10^3 because $10^8 \cdot 10^3 = 10^{11}$.
So the answer is about 1.36×10^3 gallons per day for each person.

1. Which of these methods makes more sense to you? What other method could you use?
2. Do you use more than 1,000 gallons of water each day at home and at school? What might explain such a high average water use?
3. What other questions could you answer with the data given?

continued on the next page >

Problem 5.4 *continued*

B In Illinois, 11,600,000 people got their water from public supplies provided by cities and towns. Those people used 1.05×10^9 gallons of water each day.

1. How many gallons of water per day did each person use?
2. Compare the amount of water from public supplies to the amount used per person in all of the United States. What could account for the difference?
3. Use your answer to Question B, part (1), to find the number of gallons each person in Illinois used *per second*. (There are 86,400 seconds in a day.) Write your answer in scientific notation.

C What percent of the water used in the United States each day went to each of the following? (Total use was 4.10×10^{11} gallons of water per day.)

1. irrigation (1.28×10^{11} gallons of water per day)
2. livestock (2.14×10^9 gallons of water per day)

D Use what you know about the rules of exponents and scientific notation to solve the following. Express your answer in scientific notation.

1. $(4.0 \times 10^2) \times (3.5 \times 10^3)$
2. $(2.0 \times 10^6) \div (2.5 \times 10^2)$

ACE Homework starts on page 88.

5.5 Revisiting Exponential Functions

You have studied situations that show patterns of exponential growth or exponential decay. All of these situations represent exponential functions that are modeled with equations of the form $y = a(b^x)$. In this equation, a is the starting value and b is the growth or decay factor.

- What are the effects of a and b on the graph of the equation?

Problem 5.5

You can use your graphing calculator to explore how the values of a and b affect the graph of $y = a(b^x)$.

A 1. Let $a = 1$. Make a prediction about how the value of b affects the graph.

2. Graph the four equations below in the same window. Use window settings that show x-values from 0 to 5 and y-values from 0 to 20.

$y = 1.25^x$ $\quad$ $y = 1.5^x$ $\quad$ $y = 1.75^x$ $\quad$ $y = 2^x$

What are the similarities in the graphs? What are the differences? Record your observations.

3. Next, graph the three equations below in the same window. Use window settings that show $0 \le x \le 5$ and $0 \le y \le 1$.

$y = 0.25^x$ $\quad$ $y = 0.5^x$ $\quad$ $y = 0.75^x$

Record your observations.

4. Describe how you could predict the general shape of the graph of $y = b^x$ for a specific value of b.

B Next, you will explore how the value of a affects the graph of $y = a(b^x)$. You may need to adjust the window settings as you work.

1. Make a prediction about how the value of a affects the graph.

2. Graph these equations in the same window. Record your observations.

$y = 2(1.5)^x$ $\quad$ $y = 3(1.5^x)$ $\quad$ $y = 4(1.5^x)$

3. Graph these equations in the same window. Record your observations.

$y = 2(0.5^x)$ $\quad$ $y = 3(0.5^x)$ $\quad$ $y = 4(0.5)^x$

4. Describe how the value of a affects the graph of an equation of the form $y = a(b^x)$.

C You have explored the effects of a and b in the exponential equation $y = a(b^x)$. Compare those effects to the effects of m and b in the linear equation $y = mx + b$.

ACE Homework starts on page 88.

ACE Applications | Connections | Extensions

Applications

1. Several students were working on Question A of Problem 5.1. They wondered what would happen if they extended their table. Do you agree or disagree with each conjecture below? Explain.

Heidi's conjecture:
The 1^x column will contain only ones.

Evan's conjecture:
The bottom right corner of any table will always have the largest value.

Roger's conjecture:
So far, every number in the 2 column is even. Eventually an odd number will show up if I extend the table far enough.

Jean's conjecture:
Any odd power (an odd row) will have all odd numbers in it.

Chaska's conjecture:
To get from one row to the next in the tens column multiply the number you have by 10. For example $10^5 = 100{,}000$, so $10^6 = 100{,}000 \times 10 = 1{,}000{,}000$.

Tim's conjecture:
The row where $x = 2$ will always have square numbers in it.

2. **Multiple Choice** Which expression is equivalent to $2^9 \times 2^{10}$?

   A. 2^{90} **B.** 2^{19} **C.** 4^{19} **D.** 2^{18}

Use the properties of exponents to write each expression as a single power. Check your answers.

3. $5^6 \times 8^6$
4. $(7^5)^3$
5. $\frac{8^{15}}{8^{10}}$

For Exercises 6–11, tell whether the statement is *true* or *false*. Explain.

6. $6^3 \times 6^5 = 6^8$
7. $2^3 \times 3^2 = 6^5$
8. $3^8 = 9^4$
9. $4^3 + 5^3 = 9^3$
10. $2^3 + 2^5 = 2^3(1 + 2^2)$
11. $\frac{5^{12}}{5^4} = 5^3$

12. **Multiple Choice** Which number is the ones digit of $2^{10} \times 3^{10}$?

F. 2 **G.** 4 **H.** 6 **J.** 8

For Exercises 13 and 14, find the ones digit of the product.

13. $4^{15} \times 3^{15}$

14. $7^{15} \times 4^{20}$

15. Manuela came to the following conclusion about power of 2.

> It must be true that $2^{10} = 2^4 \cdot 2^6$, because I can group
> $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ as
> $(2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$

a. Verify that Manuela is correct by evaluating both sides of the equation $2^{10} = 2^4 \cdot 2^6$.

b. Use Manuela's idea of grouping factors to write three other expressions that are equivalent to 2^{10}. Evaluate each expression you find to verify that it is equivalent to 2^{10}.

c. The standard form for 2^7 is 128, and the standard form for 2^5 is 32. Use these facts to evaluate 2^{12}. Explain your work.

d. Test Manuela's idea to see if it works for exponential expressions with other bases, such as 3^8 or $(1.5)^{11}$. Test several cases. Give an argument supporting your conclusion.

For Exercises 16–21, tell whether each expression is equivalent to 1.25^{10}. Explain your reasoning.

16. $(1.25)^5 \cdot (1.25)^5$

17. $(1.25)^3 \times (1.25)^7$

18. $(1.25) \times 10$

19. $(1.25) + 10$

20. $(1.25^5)^2$

21. $(1.25)^5 \cdot (1.25)^2$

For Exercises 22–25, tell whether each expression is equivalent to $(1.5)^7$. Explain your reasoning.

22. $1.5^5 \times 1.5^2$

23. $1.5^3 \times 1.5^4$

24. 1.5×7

25. $(1.5) + 7$

26. Some students are trying to solve problems with rational exponents. Which of these solutions is correct?

Stu's Solution

$$81^{\frac{3}{4}} = \left(81^{\frac{1}{4}}\right)^3$$
$$= (3^3)$$
$$= 27$$

Carrie's Solution

$$125^{\frac{7}{3}} = 125^{\frac{6}{3}+\frac{1}{3}}$$
$$= 125^2 \cdot 125^{\frac{1}{3}}$$
$$= 15{,}625 \cdot 5$$
$$= 78{,}125$$

For Exercises 27–30, use the properties of exponents to evaluate each expression.

27. $\left(756^{\frac{1}{7}}\right)^7$

28. $342^{\frac{5}{2}} \div 342^{\frac{3}{2}}$

29. $3^{35} \cdot 3^{-35}$

30. $\left(\frac{1}{2}\right)^{40} \cdot 2^{40}$

For Exercises 31–36, decide if each statement is *always true, always false,* or *sometimes true.* Explain.

31. $2^n \cdot 2^n = 2(2^n)$

32. $2^n \cdot 2^n = (2^n)^2$

33. 2^n is less than 2^{n-1}.

34. b^n is less than b^{n-1}.

35. For the expression 3^x, when x is negative, 3^x will be smaller than 1.

36. For the expression b^x, when x is negative, b^x will be smaller than 1.

37. In 1867, the United States of America purchased the territory of Alaska from the Russian Empire. Its 586,412 square miles cost \$7.2 million. The United States paid roughly two cents per acre of land. Assume that the price of land in Alaska has increased in value by 5% a year since the purchase.

   a. Write an equation that represents the price per acre in the year n.

   b. What was the cost of an acre in 1900? In 2000?

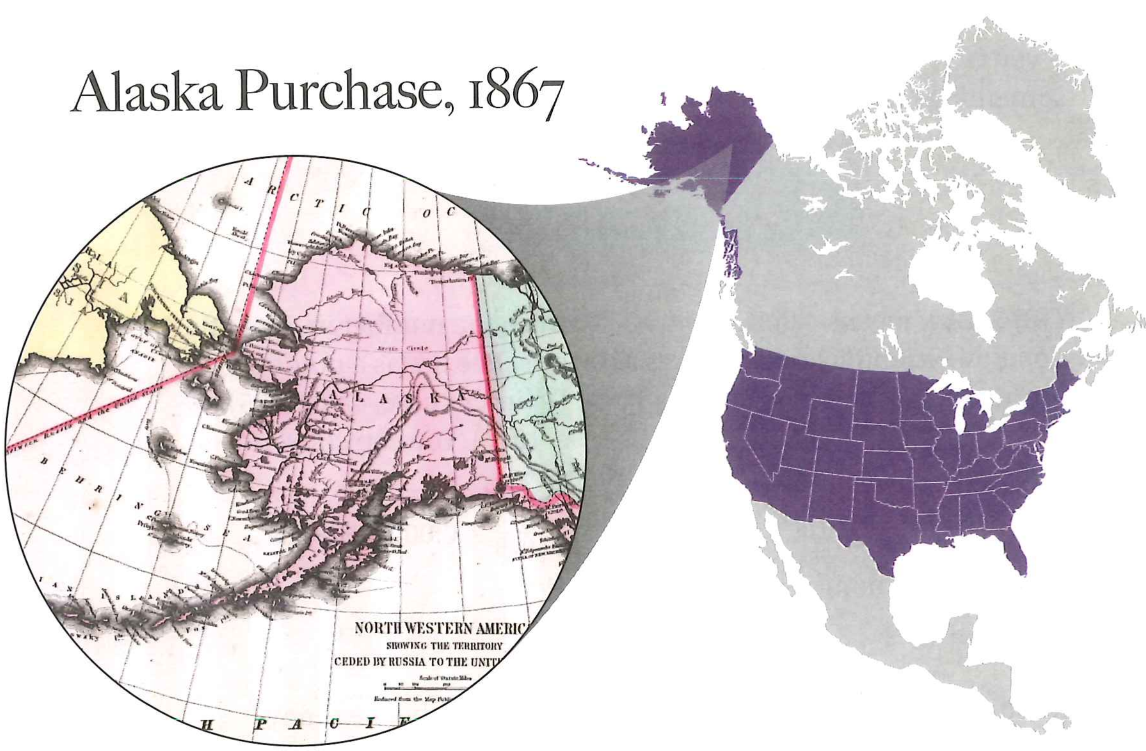
   c. In what year did the cost reach approximately \$1 per acre? \$100 per acre?

   d. Gia calculated the cost per acre after n years on her calculator. She got the answer 2.453774647E28. For what year was she trying to find the cost?

38. Suppose n is the number of years after the United States purchased the territory of Alaska, in March of 1867. The equation $v = 7{,}200{,}000 \cdot (1.05)^n$ models the total value v of the territory. It is based on a 5% increase per year. Calculate the value of the territory during each month below. Explain what exponent you would use.

   a. April 1867
   b. May 1867
   c. September 1867
   d. June 1868
   e. November 1868

Alaska Purchase, 1867



39. Copy and complete this table.

Powers of Ten

Standard Form	Exponential Form
10,000	10^4
1,000	10^3
100	10^2
10	10^1
1	10^0
$\frac{1}{10} = 0.1$	10^{-1}
$\frac{1}{100} = 0.01$	10^{-2}
$\frac{1}{1,000} = 0.001$	■
$\frac{1}{10,000} = 0.0001$	■
■	10^{-5}
■	10^{-6}

40. Write each number in standard form as a decimal.

3×10^{-1} 1.5×10^{-2} 1.5×10^{-3}

41. If you use your calculator to compute $2 \div 2^{12}$, the display might show something like this:

4.8828125E⁻4

The display means 4.8828125×10^{-4}, which is a number in scientific notation. Scientific notation uses two parts. The first is a number greater than or equal to 1 but less than 10 (in this case, 4.8828125). The second is a power of 10 (in this case, 10^{-4}). You can convert 4.8828125×10^{-4} to standard form in this way.

$$4.8828125 \times 10^{-4} = 4.8828125 \times \frac{1}{10,000} = 0.00048828125$$

a. Write each number in standard notation.

1.2×10^{-1} 1.2×10^{-2} 1.2×10^{-3} 1.2×10^{-8}

b. Suppose you have the expression 1.2×10^{-n}, where n is any whole number greater than or equal to 1. Using what you discovered in part (a), explain how you would write the expression in standard notation.

42. Write each number in scientific notation.

a. 2,000,000 **b.** 28,000,000 **c.** 19,900,000,000

d. 0.12489 **e.** 0.0058421998 **f.** 0.0010201

43. When Tia divided 0.0000015 by 1,000,000 on her calculator, she got 1.5E−12, which means 1.5×10^{-12}.

a. Write a different division problem that will give the result 1.5E−12 on your calculator.

b. Write a multiplication problem that will give the result 1.5E−12 on your calculator.

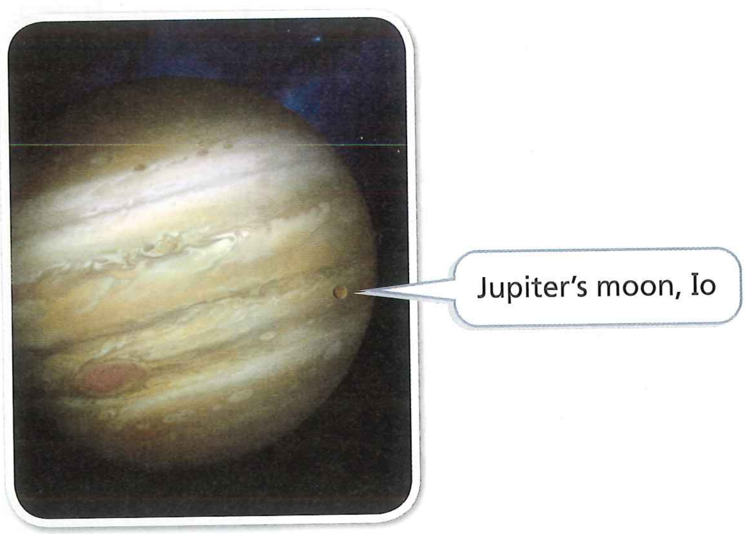
44. The radius of the moon is about 1.74×10^6 meters.

a. Express the radius of the moon in standard notation.

b. The largest circle that will fit on your textbook page has a radius of 10.795 cm. Express this radius in meters, using scientific notation.

c. Suppose a circle has the same radius as the moon. By what scale factor would you reduce the circle to fit on your textbook page?

d. Earth's moon is about the same size as Io, one of Jupiter's moons. What is the ratio of the moon's radius to the radius of Jupiter (6.99×10^7 meters)?



45. The number 2^7 is written in standard form as 128 and in scientific notation as 1.28×10^2. The number $\left(\frac{1}{2}\right)^7$, or $(0.5)^7$, is written in standard form as 0.0078125 and in scientific notation as 7.8125×10^{-3}. Write each number in scientific notation.

a. 2^8 **b.** $\left(\frac{1}{2}\right)^8$ **c.** 20^8 **d.** $\left(\frac{1}{20}\right)^8$

46. **a.** The boxes in the table below represent decreasing y-values. The decay factor for the y-values is $\frac{1}{3}$. Copy and complete the table.

x	0	1	2	3	4	5	6	7	8
y	30	10	■	■	■	■	■	■	■

b. For $x = 12$, a calculator gives a y-value of 5.645029269E−5. What does that mean?

c. Write the y-values for $x = 8, 9, 10,$ and 11 in scientific notation.

For Exercises 47–49, use the properties of exponents to show that each statement is true.

47. $\frac{1}{2}(2^n) = 2^{n-1}$

48. $4^{n-1} = \frac{1}{4}(4^n)$

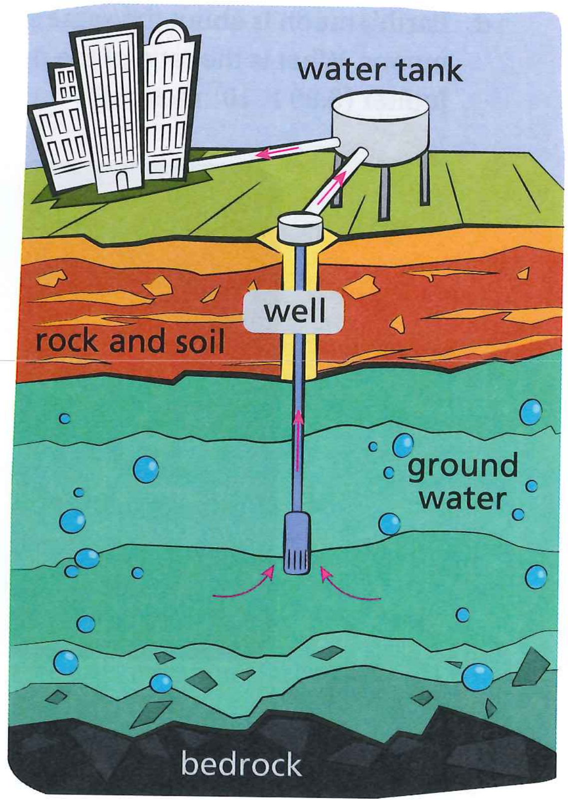
49. $25(5^{n-2}) = 5^n$

50. Use the data from Problem 5.4 to answer the following questions. Write your final answer in scientific notation.

a. How many of gallons of water are used in the United States in a year?

b. About how many times greater is the amount of water used for irrigation than the amount used for livestock?

c. Suppose 80% of water is from *surface* sources. How many gallons of freshwater are removed *from the ground* each month?



For Exercises 51–57, rewrite each expression in scientific notation.

51. $(8.2 \times 10^2) \times (2.1 \times 10^5)$

52. $(2.0 \times 10^3) \times (3.5 \times 10^6) \times (3.0 \times 10^3)$

53. $(2.0 \times 10^8) \times (1.4 \times 10^{-10})$

54. $(5.95 \times 10^8) \div (1.70 \times 10^5)$

55. $(1.28 \times 10^6) \div (5.12 \times 10^7)$

56. $(2.8 \times 10^{-4}) \div (1.4 \times 10^4)$

57. $(3.6 \times 10^2) \div (9.0 \times 10^{-3})$

For Exercises 58–62, find the missing values in each equation. Choose values such that all numbers are written in correct scientific notation.

58. $(2.4 \times 10^3) \times (g \times 10^h) = 6.0 \times 10^{12}$

59. $(j \times 10^2) \times (1.8 \times 10^k) = 9.0 \times 10^1$

60. $(m \times 10^7) \div (2.4 \times 10^n) = 5.0 \times 10^4$

61. $(6.48 \times 10^6) \div (p \times 10^q) = 2.16 \times 10^{-2}$

62. $(r \times 10^s) \times (r \times 10^s) = 1.6 \times 10^5$

63. Without actually graphing these equations, describe and compare their graphs. Be as specific as you can.

$y = 4^x$ $\quad$ $y = 0.25^x$ $\quad$ $y = 10(4^x)$ $\quad$ $y = 10(0.25^x)$

64. Explain how each of the graphs for the equations below will differ from the graph of $y = 2^x$.

a. $y = 5(2^x)$

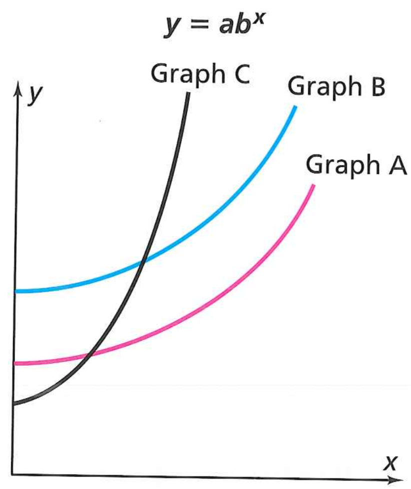
b. $y = (5 \cdot 2)^x$

c. $y = \frac{1}{2}(2^x)$

d. $y = -1(2^x)$

e. $y = \left(\frac{1}{2}\right)^x$

65. Each graph below represents an exponential equation of the form $y = a(b^x)$.



a. For which of the three functions is the value of a greatest?

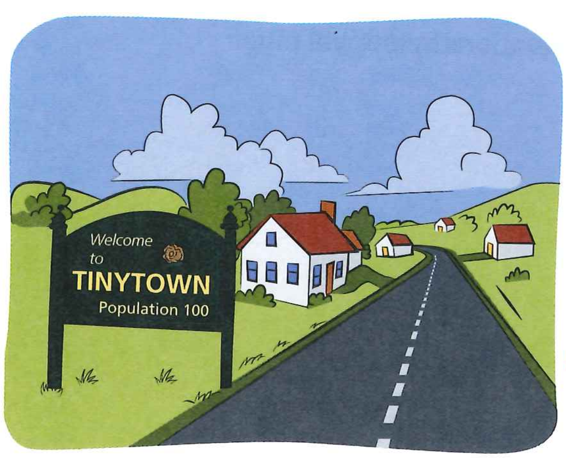
b. For which of the three functions is the value of b greatest?

66. Grandville has a population of 1,000. Its population is expected to decrease by 4% a year for the next several years. Tinytown has a population of 100. Its population is expected to increase by 4% a year for the next several years. For parts (a)–(c), explain how you found each answer.

a. What is the population of each town after 5.5 years?

b. In how many years will Tinytown have a population of approximately 1,342? Explain your method.

c. Will the populations of the two towns ever be the same? Explain.



Connections

In Exercises 67–69, tell how many zeros are in the standard form of each number.

67. 10^{10}

68. 10^{50}

69. 10^{100}

In Exercises 70 and 71, find the least integer value of x that will make each statement true.

70. $9^6 < 10^x$

71. $3^{14} < 10^x$

In Exercises 72–74, identify the greater number in each pair.

72. 6^{10} or 7^{10}

73. 8^{10} or 10^8

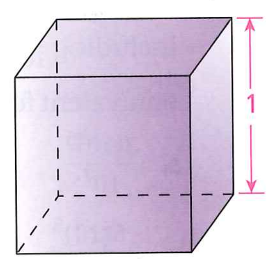
74. 6^9 or 9^6

In Exercises 75 and 76, tell whether each statement is *true* or *false*. Do not do an exact calculation. Explain your reasoning.

75. $(1.56892 \times 10^5) - (2.3456 \times 10^4) < 0$

76. $\frac{3.96395 \times 10^5}{2.888211 \times 10^7} > 1$

77. Suppose you start with a unit cube (a cube with edges of length 1 unit). In parts (a)–(c), give the volume and surface area of the cube that results from the given transformation.

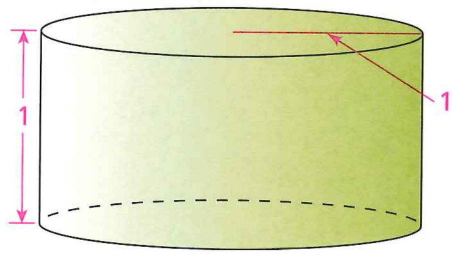


a. Each edge length is doubled.

b. Each edge length is tripled.

c. Each edge is enlarged by a scale factor of 100.

78. Suppose you start with a cylinder that has a radius of 1 unit and a height of 1 unit. In parts (a)–(c), give the volume of the cylinder that results from the given transformation.



a. The radius and height are doubled.

b. The radius and height are tripled.

c. The radius and height are enlarged by a scale factor of 100.

79. a. Tell which of the following numbers are prime. (There may be more than one.)

$2^2 - 1$ $\quad$ $2^3 - 1$ $\quad$ $2^4 - 1$ $\quad$ $2^5 - 1$ $\quad$ $2^6 - 1$

b. Find another prime number that can be written in the form $2^n - 1$.

80. In parts (a)–(d), find the sum of the proper factors for each number.

a. 2^2

b. 2^3

c. 2^4

d. 2^5

e. What do you notice about the sums in parts (a)–(d)?

81. The expression $\frac{20}{10^2}$ can be written in many equivalent forms, including $\frac{2}{10}$, $\frac{1}{5}$, 0.2, and $\frac{2(10^2)}{10^3}$. In parts (a) and (b), write two equivalent forms for each expression.

a. $\frac{3(10)^5}{10^7}$

b. $\frac{5(10)^5}{25(10)^7}$

Extensions

In Exercises 82–86, predict the ones digit for the standard form of each number.

82. 7^{100}

83. 6^{200}

84. 17^{100}

85. 31^{10}

86. 12^{100}

For Exercises 87 and 88, find the value of *a* that makes each number sentence true.

87. $a^7 = 823{,}543$

88. $a^6 = 1{,}771{,}561$

89. Explain how you can use your calculator to find the ones digit of the standard form of 3^{30}.

90. **Multiple Choice** In the powers table you completed in Problem 5.1, look for patterns in the ones digit of square numbers. Which number is *not* a square number? Explain.

A. 289 **B.** 784 **C.** 1,392 **D.** 10,000

91. **a.** Find the sum for each row in the table below.

Sums of Powers of $\frac{1}{2}$

Row 1	$\frac{1}{2}$
Row 2	$\frac{1}{2} + \left(\frac{1}{2}\right)^2$
Row 3	$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$
Row 4	$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

b. Study the pattern. Suppose the pattern continues. Write the expression that would be in row 5 and evaluate the sum.

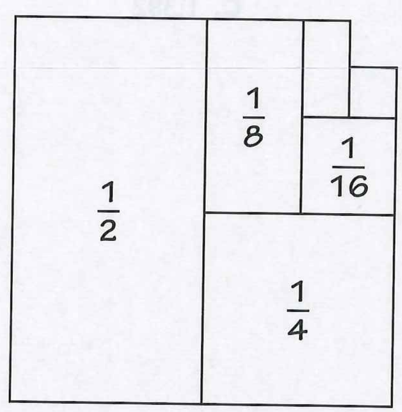
c. What would be the sum of the expression in row 10? What would you find if you evaluated the sum for row 20?

d. Describe the pattern of sums in words and with a symbolic expression.

e. For which row does the sum first exceed 0.9?

f. As the row number increases, the sum gets closer and closer to what number?

g. Celeste claims the pattern is related to the pattern of the areas of the ballots cut in Problem 4.1. She drew the picture below to explain her thinking.



row 6 = $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6$

What relationship do you think she has observed?

92. Chen, from Problem 4.1, decides to make his ballots starting with a sheet of paper with an area of 1 square foot.

a. Copy and extend this table to show the area of each ballot after each of the first 8 cuts.

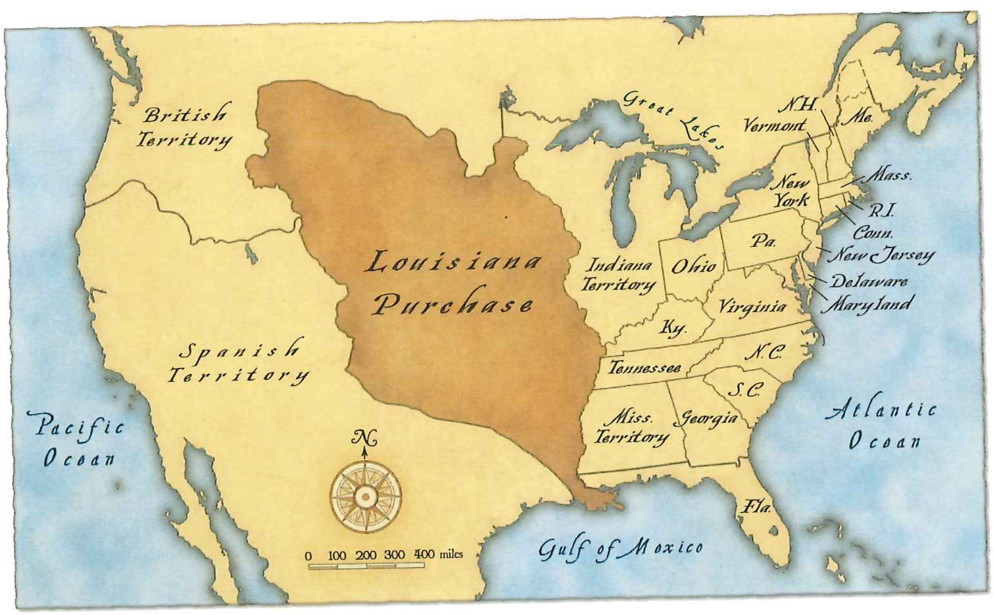
Areas of Ballots

Number of Cuts	Area (ft^2)
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

b. Write an equation for the area A of a ballot after any cut n.

c. Use your equation to find the area of a ballot after 20 cuts. Write your answer in scientific notation.

93. In 1803, the United States bought the 828,000-square-mile Louisiana Purchase for $15,000,000. Suppose one of your ancestors was given 1 acre of the Louisiana Purchase. Assuming an annual increase in value of 4%, what was the value of this acre in 2003? (640 acres = 1 square mile)



Mathematical Reflections 5

In this Investigation, you explored properties of exponents. You also looked at how the values of *a* and *b* affect the graph of $y = a(b^x)$. You made use of scientific notation to find relations among very large numbers. The following questions will help you summarize what you have learned.

Think about these questions. Discuss your ideas with other students and your teacher. Then write a summary of your findings in your notebook.

1. a. **Describe** some of the rules for operating with exponents.

   b. **What** is scientific notation? **What** are its practical applications?

2. **Describe** the effects of *a* and *b* on the graph of $y = a(b^x)$.

3. **Compare** exponential and linear functions. Include in your comparison information about their patterns of change, *y*-intercepts, whether the function is decreasing or increasing, and any other information you think is important. Include examples of how they are useful.

Common Core Mathematical Practices

As you worked on the Problems in this Investigation, you used prior knowledge to make sense of them. You also applied Mathematical Practices to solve the Problems. Think back over your work, the ways you thought about the Problems, and how you used Mathematical Practices.

Hector described his thoughts in the following way:

In Problem 5.4, we noticed that it is easy to use the rules of exponents to do multiplication when large numbers are expressed in scientific notation. Most of our group used a calculator anyway.

Since we were working with approximate data, we knew that our answers were also approximate.

We all used the graphing calculator in Problem 5.5. The calculator was faster for making the graphs so we could compare families of exponential functions.

Common Core Standards for Mathematical Practice

MP5 Use appropriate tools strategically

- What other Mathematical Practices can you identify in Hector's reasoning?
- Describe a Mathematical Practice that you and your classmates used to solve a different Problem in this Investigation.

Unit Project

Half-Life

Most things around you are composed of atoms that are stable. However, the atoms that make up *radioactive* substances are unstable. They break down in a process known as *radioactive decay*. From their decay, they emit radiation. At high levels, radiation can be dangerous.

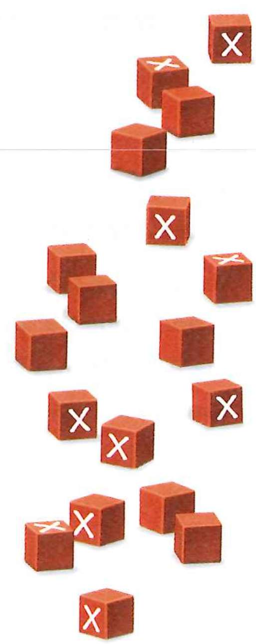
Rates of decay vary from substance to substance. The term *half-life* describes the time it takes for half of the atoms in a radioactive sample to change into other more stable atoms. For example, the half-life of carbon-11 is 20 minutes. This means that 2,000 carbon-11 atoms are reduced to 1,000 carbon-11 atoms and 1,000 boron-11 atoms in 20 minutes. After 40 minutes, the carbon-11 atoms are reduced to 500 carbon-11 atoms and 1,500 boron-11 atoms.

Half-lives vary from a fraction of a second to billions of years. For example, the half-life of polonium-214 is 0.00016 seconds. The half-life of rubidium-87 is 49 billion years.

In this experiment, you will model the decay of a radioactive substance known as iodine-124. About $\frac{1}{6}$ of the atoms in a sample of iodine-124 decay each day. This experiment will help you determine the half-life of this substance.

Follow these steps to conduct your experiment:

- Use 100 cubes to represent 100 iodine-124 atoms. Mark one face of each cube.
- For the first day, place all 100 cubes in a container, shake the container, and pour the cubes onto the table.
- The cubes for which the mark is facing up represent atoms that have decayed. Remove these cubes, and record the number of cubes that remain.
- For the next day, place the remaining cubes in the container, shake the container, and pour the cubes onto the table.
- Repeat the last two steps until one cube or no cubes remain.



When you complete your experiment, answer the following questions.

1. a. In your experiment, how many days did it take to reduce the 100 iodine-124 atoms to 50 atoms? In other words, how many times did you have to roll the cubes until about 50 cubes remained?

   b. How many days did it take to reduce 50 iodine-124 atoms to 25 atoms?

   c. Based on your answers to parts (a) and (b), what is the half-life of iodine-124?

2. a. In a sample of real iodine-124, $\frac{1}{6}$ of the atoms decay after 1 day. What fraction of the atoms remain after 1 day?

   b. Suppose a sample contains 100 iodine-124 atoms. Use your answer from part (a) to write an equation for the number of atoms n remaining in the sample after d days.

   c. Use your equation to find the half-life of iodine-124.

   d. How does the half-life you found based on your equation compare to the half-life you found from your experiment?

3. a. Make up a problem involving a radioactive substance with a different rate of decay that can be modeled by an experiment involving cubes or other common objects. Describe the situation and your experiment.

   b. Conduct your experiment and record your results.

   c. Use your results to predict the half-life of your substance.

   d. Use what you know about the rate of decay to write an equation that models the decay of your substance.

   e. Use your equation to find the half-life of your substance.

Write a report that summarizes your findings about decay rates and half-lives. Your report should include tables and graphs justifying your answers to the questions above.

Looking Back

You developed your skills in recognizing and applying exponential relationships between variables by working on Problems in this Unit.

You wrote equations of the form $y = a(b^x)$ to describe exponential growth of populations and investments and exponential decay of medicines and radioactive materials. You used equations to produce tables and graphs of the relationships. You used those tables and graphs to make predictions and solve equations.

Use Your Understanding: Algebraic Reasoning

To test your understanding and skill in finding and applying exponential models, solve these problems. These problems arise as the student council at Lincoln Middle School plans a fundraising event.

The students want to have a quiz show called *Who Wants to Be Rich?* Contestants will be asked a series of questions. A contestant will play until he or she misses a question. The total prize money will grow with each question answered correctly.

1. Lucy proposes that a contestant receive \$5 for answering the first question correctly. For each additional correct answer, the total prize would increase by \$10.
   a. For Lucy's proposal, what equation gives the total prize p for correctly answering n questions?
   b. How many questions would a contestant need to answer correctly to win at least \$50? To win at least \$75? To win at least \$100?
   c. Sketch a graph of the (n, p) data for $n = 1$ to 10.

2. Armando also thinks the first question should be worth $5. However, he thinks a contestant's winnings should double with each subsequent correct answer.

   a. For Armando's proposal, what equation gives the total prize p for correctly answering n questions?

   b. How many questions will a contestant need to answer correctly to win at least $50? To win at least $75? To win at least $100?

   c. Sketch a graph of the data (n, p) for $n = 1$ to 10.

3. The council decides that contestants for *Who Wants to Be Rich?* will be chosen by a random drawing. Students and guests at the fundraiser will buy tickets like the one below.

The purchaser will keep half of the ticket and add the other half to the entries for the drawing.

   a. To make the tickets, council members will take a large piece of paper and fold it in half many times to make a grid of small rectangles. How many rectangles will there be after n folds?

   b. The initial piece of paper will be a square with sides measuring 60 centimeters. What will be the area of each rectangle after n folds?

Decide whether each statement is *true* or *false*. Explain.

4. $3^5 \times 6^5 = 9^5$

5. $8^5 \times 4^6 = 2^{27}$

6. $\frac{2^0 \times 6^7}{3^7} = 2^7$

7. $8^{\frac{3}{2}} \times 2^{\frac{1}{2}} = 32$

8. $1.39 \times 10^{-5} = 139{,}000$

9. $1.099511 \times 10^6 = 1{,}099{,}511$

Explain Your Reasoning

To answer Questions 1–3, you had to use algebraic knowledge about number patterns, graphs, and equations. You had to recognize linear and exponential patterns from verbal descriptions and represent those patterns with equations and graphs.

10. How can you decide whether a data pattern can be modeled by an exponential equation of the form $y = a(b^x)$? How will the values of a and b relate to the data pattern?

11. Describe the possible shapes for graphs of exponential relationships. How can the shape of an exponential graph be predicted from the values of a and b in the equation?

12. How are the data patterns, graphs, and equations for exponential relationships similar to those for linear relationships? How are they different?

13. Describe the rules for exponents that you used in Questions 4–9. Choose one of the rules and explain why it works.

English / Spanish Glossary

B

base The number that is raised to a power in an exponential expression. In the expression 3^5, read "3 to the fifth power", 3 is the base and 5 is the exponent.

base El número que se eleva a una potencia en una expresión exponencial. En la expresión 3^5, que se lee "3 elevado a la quinta potencia", 3 es la base y 5 es el exponente.

C

compound growth Another term for exponential growth, usually used when talking about the monetary value of an investment. The change in the balance of a savings account shows compound growth because the bank pays interest not only on the original investment, but on the interest earned.

crecimiento compuesto Otro término para crecimiento exponencial, normalmente usado para referirse al valor monetario de una inversión. El cambio en el saldo de una cuenta de ahorros muestra un crecimiento compuesto, ya que el banco paga intereses no sólo sobre la inversión original, sino sobre los intereses ganados.

D

decay factor The constant factor that each value in an exponential decay pattern is multiplied by to get the next value. The decay factor is the base in an exponential decay equation, and is a number between 0 and 1. For example, in the equation $A = 64(0.5)^n$, where A is the area of a ballot and n is the number of cuts, the decay factor is 0.5. It indicates that the area of a ballot after any number of cuts is 0.5 times the area after the previous number of cuts. In a table of (x, y) values for an exponential decay relationship (with x-values increasing by 1), the decay factor is the ratio of any y-value to the previous y-value.

factor de disminución El factor constante por el cual se multiplica cada valor en un patrón de disminución exponencial para obtener el valor siguiente. El factor de disminución es la base en una ecuación de disminución exponencial. Por ejemplo, en la ecuación $A = 64(0.5)^n$, donde A es el área de una papeleta y n es el número de cortes, el factor de disminución es 0.5. Esto indica que el área de una papeleta después de un número cualquiera de cortes es 0.5 veces el área después del número anterior de cortes. En una tabla de valores (x, y) para una relación de disminución exponencial (donde el valor x crece de a 1), el factor de disminución es la razón entre cualquier valor de y y su valor anterior.

decay rate The percent decrease in an exponential decay pattern. In general, for an exponential pattern with decay factor b, the decay rate is $1 - b$.

tasa de disminución El porcentaje de reducción en un patrón de disminución exponencial. En general, para un patrón exponencial con factor de disminución b, la tasa de disminución es $1 - b$.

decide Academic Vocabulary
To use the given information and any related facts to find a value or make a determination.

related terms *determine, find, conclude*

sample Study the pattern in the table. Decide whether the relationship is linear or exponential.

x	−1	0	1	2	3
y	−9	−7	−5	−3	−1

Each y-value Increases by 2 when each x-value Increases by 1. The relationship is linear.

decidir Vocabulario academico
Usar la información dada y los datos relacionados para hallar un valor o tomar una determinación.

terminos relacionados *decidir, hallar, calcular, concluir*

ejemplo ?Cual es una manera de determinar la descomposicion en factores primos de 27?

x	−1	0	1	2	3
y	−9	−7	−5	−3	−1

Cada valor de y aumenta en 2 cuando cada valor de x aumenta en 1. La relaclón es lineal.

describe Academic Vocabulary
To explain or tell in detail. A written description can contain facts and other information needed to communicate your answer. A diagram or a graph may also be included when you describe something.

related terms *explain, tell, present, detail*

sample Consider the following equations:

Equation 1 $y = 3x + 5$

Equation 2 $y = 5\left(3^x\right)$

Use a table to describe the change in y-values as the x-values increase in both equations.

x	0	1	2	3	4
$y = 3x + 5$	5	8	11	14	17
$y = 5(3^x)$	5	15	45	135	405

In $y = 3x + 5$, the value of y increases by 3 when x increases by 1. In $y = 5(3^x)$, the value of y increases by a factor of 3 when x increases by 1.

describir Vocabulario academico Explicar usando detalles. Puedes describir una situacion usando palabras, numeros, graficas, tablas o cualquier combinacion de estos.

terminos relacionados *explicar, decir, presentar, dar detalles*

ejemplo Considera las siguintes ecuaciones.

Ecuacion 1 $y = 3x + 5$

Ecuacion 2 $y = 5\left(3^x\right)$

Usa una tabla para describir el cambio en los valores de y a medida que los valores de x se incrementan en ambas ecuaciones.

x	0	1	2	3	4
$y = 3x + 5$	5	8	11	14	17
$y = 5(3^x)$	5	15	45	135	405

In $y = 3x + 5$, el valor de y aumenta en 3 cuando x aumenta en 1. En $y = 5(3^x)$, el valor de y aumenta por un factor de 3 cuando x aumenta en 1.

E

explain Academic Vocabulary
To give facts and details that make an idea easier to understand. Explaining can involve a written summary supported by a diagram, chart, table, or any combination of these.

related terms *describe, justify, tell*

sample Etymologists are working with a population of mosquitoes that have a growth factor of 8. After 1 month there are 6,000 mosquitoes. In two months, there are 48,000 mosquitoes.

Write an equation for the population after any number of months. Explain each part of your equation.

> I first find the initial population of mosquitoes by dividing 6,000 by 8 to get 750. I can then model the population growth with the equation $y = 750(8^m)$, where 750 represents the initial population, 8 is the growth factor, *m* is the number of months, and *y* is the population of mosquitoes after *m* months.

explicar Vocabulario academico
Dar hechos *y* detalles que hacen que una idea sea mas facil de comprender. Explicar puede implicar un resumen escrito apoyado por un diagrama, un grafica, una table o cualquier combatinacion de estos.

terminos relacionados *describir, justificar, decir*

ejemplo Los entomologos trabajan con una poblacion de mosquitos que tiene un factor de crecimiento de 8. Despues de 1 mes hay 6,000 mosquitos. En dos meses, hay 48,000 mosquitos.

Escribe una ecuacion para la poblacion despues de cualquier numero de meses. Explica cada parte de tu ecuacion.

> Primero hallo la población inicial de mosquitos dividiendo 6,000 entre 8 para obtener 750. Luego puedo modelar el crecimiento de la población con la ecuación $y = 750(8^m)$, donde 750 representa la población inicial, 8 es el factor de crecimiento, *m* es el número de meses y *y* es la población de mosquitos luego de *m* meses.

---

exponent The small raised number that tells how many times a factor is used. For example, 5^3 means $5 \times 5 \times 5$. The 3 is the exponent.

exponente El pequeño número elevado que dice cuántas veces se usa un factor. Por ejemplo, 5^3 significa $5 \times 5 \times 5$. El 3 es el exponente.

---

exponential decay A pattern of decrease in which each value is found by multiplying the previous value by a constant factor greater than 0 and less than 1. For example, the pattern 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, . . . shows exponential decay in which each value is $\frac{1}{3}$ times the previous value.

disminución exponencial Un patrón de disminución en el cual cada valor se calcula multiplicando el valor anterior por un factor constante mayor que 0 y menor que 1. Por ejemplo, el patrón 27, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, . . . muestra una disminución exponencial en la que cada valor es $\frac{1}{3}$ del valor anterior.

---

exponential form A quantity expressed as a number raised to a power. In exponential form, 32 can be written as 2^5.

forma exponencial Una cantidad que se expresa como un número elevado a una potencia. En forma exponencial, 32 puede escribirse como 2^5.

exponential functions Relationships between two variables that are exponential. For example, the function represented by $y = 4^{n-1}$ for placing 1 ruba on square one, 4 rubas on square two, 16 rubas on square three, and so on, is an exponential function.

funciones exponenciales Relaciones entre dos variables que son exponenciales. Por ejemplo, la función representada por $y = 4^{n-1}$ para poner un ruba en el cuadro uno, cuatro rubas en el cuadro dos, dieciséis rubas en el cuadro tres y así sucesivamente, es una función exponencial.

exponential growth A pattern of increase in which each value is found by multiplying the previous value by a constant factor greater than 1. For example, the doubling pattern 1, 2, 4, 8, 16, 32, . . . shows exponential growth in which each value is 2 times the previous value.

crecimiento exponencial Un patrón de crecimiento en el cual cada valor se calcula multiplicando el valor anterior por un factor constante mayor que 1. Por ejemplo, el patrón 1, 2, 4, 8, 16, 32, . . . muestra un crecimiento exponencial en el que cada valor es el doble del valor anterior.

exponential relationship A relationship that shows exponential growth or decay.

relación exponencial Una relación que muestra crecimiento o disminución exponencial.

G

growth factor The constant factor that each value in an exponential growth pattern is multiplied by to get the next value. The growth factor is the base in an exponential growth equation, and is a number greater than 1. For example, in the equation $A = 25(3)^d$, where A is the area of a patch of mold and d is the number of days, the growth factor is 3. It indicates that the area of the mold for any day is 3 times the area for the previous day. In a table of (x, y) values for an exponential growth relationship (with x-values increasing by 1), the growth factor is the ratio of any y-value to the previous y-value.

factor de crecimiento El factor constante por el cual se multiplica cada valor en un patrón de crecimiento exponencial para obtener el valor siguiente. El factor de crecimiento es la base en una ecuación de crecimiento exponencial. Por ejemplo, en la ecuación $A = 25(3)^d$, donde A es el área enmohecida y d es el número de días, el factor de crecimiento es 3. Esto indica que el área enmohecida en un día cualquiera es 3 veces el área del día anterior. En una tabla de valores (x, y) para una relación de crecimiento exponencial (donde el valor de x aumenta de a 1), el factor exponencial es la razón entre cualquier valor de y y su valor anterior.

growth rate The percent increase in an exponential growth pattern. For example, in Problem 3.1, the number of rabbits increased from 100 to 180 from year 0 to year 1, an 80% increase. From year 1 to year 2, the number of rabbits increased from 180 to 324, an 80% increase. The growth rate for this rabbit population is 80%. Interest, expressed as a percent, is a growth rate. For an exponential growth pattern with a growth factor of b, the growth rate is $b - 1$.

tasa de crecimiento El porcentaje de crecimiento en un patrón de crecimiento exponencial. Por ejemplo, en el Problema 3.1, el número de conejos aumentó de 100 a 180 del año 0 al año 1, un aumento del 80%. Del año 1 al año 2, el número de conejos aumentó de 180 a 324, un aumento del 80%. La tasa de crecimiento para esta población de conejos es del 80%. El interés, expresado como porcentaje, es una tasa de crecimiento. Para un patrón de crecimiento exponencial con un factor de crecimiento b, la tasa de crecimiento es $b - 1$.

N

***n*th root** The nth root of a number b is a number r which, when raised to the power of n, is equal to b. That is, $r^n = b$ and $\sqrt[n]{b} = b^{\frac{1}{n}} = r$.

raíz enésima La raíz enésima de un número b es un número erre que, cuando se eleva a la potencia n, es igual a b. Es decir, y $r^n = b$ a un $\sqrt[n]{b} = b^{\frac{1}{n}} = r$.

P

predict Academic Vocabulary
To make an educated guess based on the analysis of real data.

related terms *estimate, expect*

sample Predict the ones digit for the expression 3^{11}.

3^1	3
3^2	9
3^3	27
3^4	81
3^5	243
3^6	729
3^7	2187
3^8	6561

The pattern for the ones digit of the powers of 3 is 3, 9, 7, 1, as the exponent increases by 1. If I continue the pattern, 3^9 will end with a 3, 3^{10} will end with a 9, and 3^{11} will end with a 7.

predecir Vocabulario academico
Hacer una conjectura informada basada en el analisis de datos reales.

terminos relacionados *estimar, esperar*

ejemplo Predice el digito de las unidades para la expresion 3^{11}.

3^1	3
3^2	9
3^3	27
3^4	81
3^5	243
3^6	729
3^7	2187
3^8	6561

El patrón para el dígito de las unidades de las potencias de 3 es 3, 9, 7, 1, a medida que el exponente aumenta en 1. Si continúo el patrón, 3^9 terminará con un 3, 3^{10} terminará con un 9, y 3^{11} terminará con un 7.

S

scientific notation A short way to write very large or very small numbers. A number is in scientific notation if it is of the form $a \times 10^n$, where n is an integer and $1 \leq a < 10$.

notación científica Una manera corta de escribir números muy grandes o muy pequeños. Un número está e notación científica si está en la forma $a \times 10^n$, donde n es un entero y $1 \leq a < 10$.

standard form The most common way we express quantities. For example, 27 is the standard form of 3^3.

forma normal La manera más común de expresar una cantidad. Por ejemplo, 27 es la forma normal de 3^3.

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