Applications

1. **Areas of Ballots**

Number of Cuts	Area (in.^{2})
0	324
1	162
2	81
3	40.5
4	20.25
5	10.125
6	5.0625
7	2.53125
8	1.265625
9	0.6328125
10	0.31640625

a. $A = 324\left(\frac{1}{2}\right)^n$

b. 9 cuts

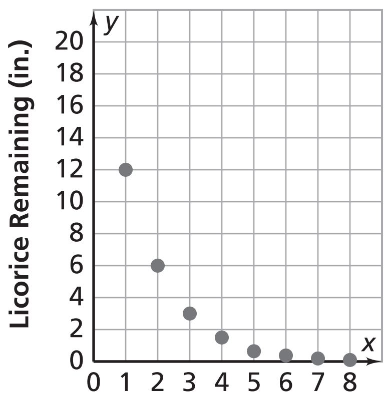
c. If the paper were at least 4,096 in.^{2}, he would be able to make 12 cuts: $1 \cdot 2^{12} = 4{,}096$.

2. Both arguments are correct. In an exponential function situation, an exponent of zero is defined for some value (the initial value). However, in an inverse variation situation, because k is chosen as some nonzero value, if $x = 0$, the equation $yx = k$ does not have a solution for y. The students may have made this conjecture because the general shape of the graph of an inverse variation situation can look like the graph of an exponential function. The second argument is also valid. In an exponential relationship, the two variables do not multiply together to give a constant. In an inverse variation, the two variables have a "factor-pair" relationship as seen in the equation $xy = k$, where k is a constant. Students might have made this conjecture since the graph of the exponential function is a curve that decreases from left to right.

3. a. **Latisha's Licorice**

Number of Friends	Licorice Remaining (in.)
1	12
2	6
3	3
4	1.5
5	0.75
6	0.375
7	0.1875
8	0.09375

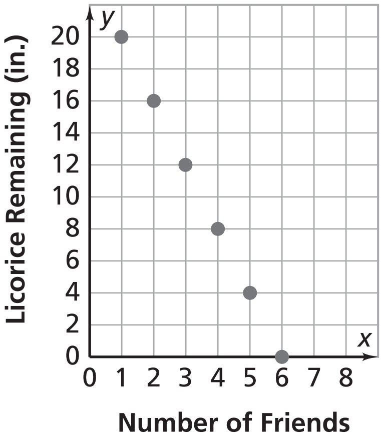
b. **Latisha's Licorice**



c. **Latisha's Licorice**

Number of Friends	Licorice Remaining (in.)
1	20
2	16
3	12
4	8
5	4
6	0

Latisha's Licorice



d. The first graph shows exponential decay; Latisha gave away less and less to each friend. The second graph is linear; each of the first six friends received the same amount. In the first graph, Latisha's licorice never runs out. In the second graph, the licorice runs out after 6 friends.

4. a. **Amount of Penicillin**

Days Since Dose	Penicillin in Blood (mg)
0	300
1	180
2	108
3	64.8
4	38.9
5	23.3
6	14.0
7	8.4

b. $d = 300(0.6)^m$

c. $d = 400(0.6)^m$, assuming the decay factor remains the same.

5. Exponential growth because $2.1 > 1$.

6. Exponential decay because $0.5 < 1$.

7. a. The decay factor is $\frac{1}{3}$ and the y-intercept is 300.

b. $y = 300\left(\frac{1}{3}\right)^x$

8. equation: $y = 24\left(\frac{1}{4}\right)^x$; decay factor: $\frac{1}{4}$; decay rate: $\frac{3}{4}$

9. equation: $y = 128\left(\frac{3}{4}\right)^x$; decay factor: $\frac{3}{4}$; decay rate: $\frac{1}{4}$

10. When $x = 1$, $y = 30$. Using the exponential equation $y = a(b)^x$, substitute the y-intercept 90 for a. Use the ordered pair and substitute 2 for x and 10 for y. Solve for b, the decay factor. The decay factor is $\frac{1}{3}$, so the equation is $y = 90\left(\frac{1}{3}\right)^x$. Substitute 1 for x to find y.

11. When $x = 4$, $y = \frac{5}{2}$. Using the exponential equation $y = a(b)^x$, substitute the y-intercept 40 for a. Use the ordered pair and substitute 2 for x and 10 for y. Solve for b, the decay factor. The decay factor is $\frac{1}{2}$, so the equation is $y = 40\left(\frac{1}{2}\right)^x$. Substitute 4 for x to find y.

12. When $x = -2$, $y = 1875$. Using the exponential equation $y = a(b)^x$, substitute the y-intercept 75 for a. Use the ordered pair and substitute 2 for x and 3 for y. Solve for b, the decay factor. The decay factor is $\frac{1}{5}$, so the equation is $y = 75\left(\frac{1}{5}\right)^x$. Substitute -2 for x to find y. **Note:** Students do not yet know that b^{-2} is the same as $\frac{1}{b^2}$, but they can still make sense of the negative exponent graphically.

13. When $x = 2$, $y = 0.64$. Using the exponential equation $y = a(b)^x$, substitute the y-intercept 64 for a. Use the ordered pair and substitute 3 for x and 0.064 for y. Solve for b, the decay factor. The decay factor is 0.1, so the equation is $y = 64(0.1)^x$. Substitute 2 for x to find y.

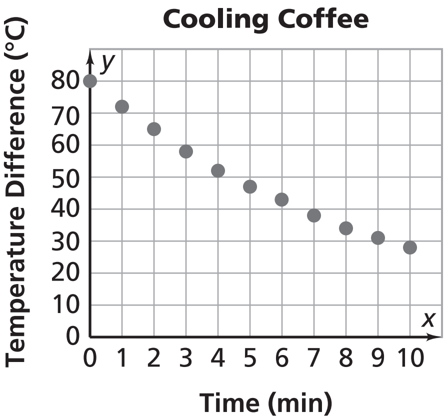
14. a. Karen thought her coupons would be used together (5% + 5% = 10% off), but that is not how coupons work. She received 5% off her original bill ($50), which made her bill $47.50. The cashier applied the next 5%-off coupon to this amount, not the original $50, so Karen's bill is $45.13.

b. $a = 50(0.95)^c$, where a represents the total amount Karen will spend and c represents the number of coupons she uses

c. \$29.94

d. Students should have an answer in the 160–180 range. If Karen uses 160 coupons, her bill is \$0.014. If she uses 170 coupons, her bill is \$0.008. 180 coupons produces a cost of \$0.00489 which would not round up to \$0.01. (**Note:** You may wish to discuss with your students whether the groceries would ever really be free (with cost equal to \$0).)

15. a.



There is a slight curve in the graph, suggesting that the temperature dropped a bit more rapidly just after it was poured. The differences between the first several pairs of temperatures in the table reflect this pattern.

b. Averaging the ratios between successive temperature differences gives a decay factor of $(0.90 + 0.90 + 0.89 + 0.90 + 0.90 + 0.91 + 0.88 + 0.89 + 0.91 + 0.90) \div 10 \approx 0.90$.

c. $d = 80(0.90)^n$, where d is temperature difference and n is time in minutes.

d. Theoretically, if the temperature decline followed an exponential pattern, the temperature would never exactly equal room temperature. However, the difference between coffee temperature and room temperature would have been less than 1°C after 42 minutes: $d = 80(0.90)^{42} = 0.96°C$

16. a. circumference $= \pi d = 5\pi \approx 15.7$ in, area $= \pi r^2 = 6.25\pi \approx 19.6$ in.^{2}

b. **Note:** Students may round answers in different ways and at different stages. This is a good opportunity to have a discussion about rounding.

Advertisement Pizza Sizes

Reduction Number	Diameter (in.)	Circumference (in.)	Area (in.^{2})
0	5.0	15.71	19.63
1	4.5	14.14	15.9
2	4.05	12.72	12.88
3	3.65	11.47	10.46
4	3.28	10.3	8.45
5	2.95	9.27	6.83

c. diameter $= 5(0.9)^n$;

circumference $= 15.7(0.9)^n$;

area $= 19.6(0.81)^n$

d. diameter $= 5(0.75)^n$;

circumference $= 15.7(0.75)^n$;

area $= 19.6(0.5625)^n$

e. $0.75 = \frac{3}{4}$; $0.5625 = \frac{9}{16}$

f. Possible answer: Yes; you can represent a 10% reduction by the expression $x - 0.10x$; you can represent 90% of original size by $0.9x$. These expressions are equivalent. (**Note:** Common language is somewhat ambiguous about the meaning of "reduction in size." If you mean reduction in dimensions, the reasoning above applies. If you mean reduction in area, it does not apply.)

17. a. 0.8; this is less than 0.9, so its product with any number will be less than the product of the same number and 0.9.

b. $\frac{2}{10}$ $\frac{2}{9}$ $(0.8)^4$ $(0.9)^4$ $(0.9)^2$ 0.84 90%

18. a. Yes, to find the growth factor of an exponential function, you divide the y-value by the previous y-value. $\frac{y_2}{y_1}$ is the growth (or decay) factor for each $(x_2 - x_1)$ unit. This is similar to defining linear growth rate, in that you need to have a change of 1 unit in x-values.

b. No, for linear relationships the growth factor is the slope, which you find by computing the change in the y-values divided by the change in the x-values.

Connections

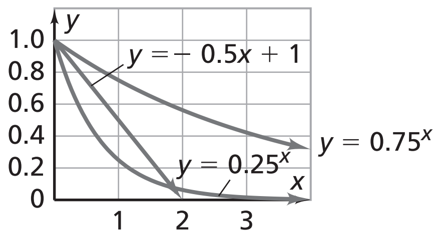
19. molecules: 3.34×10^{22}

20. red blood cells: 2.5×10^{13}

21. Earth to sun: 9.3×10^{7} mi; 1.5×10^{8} km

22. size of Milky Way: 1.0×10^{5} years; number of stars: 3.0×10^{11}

23. a.



b. The three graphs intersect at the point (0, 1). The graphs of $y = -0.5x + 1$ and $y = (0.25)^x$ also intersect at about (1.85, 0.075). In Quadrant II, there is a point of intersection for $y = -0.5x + 1$ and $y = (0.75)^x$.

c. The graph of $y = (0.25)^x$ decreases faster than that of $y = -0.5x + 1$ until about $x = 0.7$. The graph of $y = -0.5x + 1$ decreases the fastest for x-values greater than 0.7.

d. Because the graph of $y = -0.5x + 1$ is a straight line, it is not an example of exponential decay.

e. The equation $y = -0.5x + 1$ does not include a variable exponent, so it is not an example of exponential decay.

24. a.

Hop	Location
1	$\frac{1}{2}$
2	$\frac{3}{4}$
3	$\frac{7}{8}$
4	$\frac{15}{16}$
5	$\frac{31}{32}$
6	$\frac{63}{64}$
7	$\frac{127}{128}$
8	$\frac{255}{256}$
9	$\frac{511}{512}$
10	$\frac{1{,}023}{1{,}024}$

b. $1 - \left(\frac{1}{2}\right)^n$ or $\frac{2^n - 1}{2^n}$

c. No, the numerator is always 1 less than the denominator. This means that the fraction approaches, but never reaches, 1.

Extensions

25. Note: A table is helpful for answering these questions. Also, this would be a good time for students to learn how to display an answer in fractional form on their calculators. The decimal form of $\left(\frac{2}{3}\right)^5$ is 0.1316872428, which is not very helpful when one is looking for patterns. To convert a displayed decimal to a fraction on a TI-73, enter the decimal and then press $F \longleftrightarrow D$ and then ENTER.

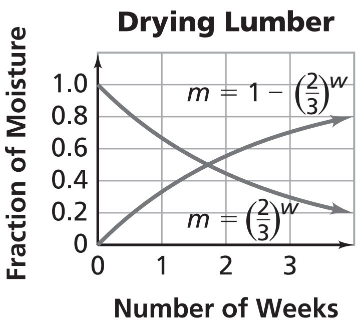
a. $\frac{32}{243}$

b. $1 - \frac{32}{243} = \frac{211}{243}$

c. $m = \left(\frac{2}{3}\right)^w$

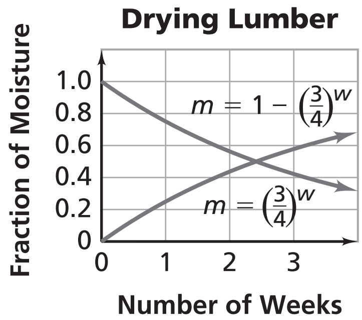
d. $m = 1 - \left(\frac{2}{3}\right)^w$

e. The graphs are mirror images of each other across the line $y = 0.5$. One approaches the x-axis, showing that the moisture remaining approaches 0; the other approaches the line $y = 1$, showing that the moisture removed approaches 100%



f. moisture remaining $= \left(\frac{3}{4}\right)^w$; moisture removed $= 1 - \left(\frac{3}{4}\right)^w$

g. These graphs are also mirror images across the line $y = 0.5$. They are stretched out farther to the right, which indicates that the moisture removal proceeds more slowly.



h. Possible answer: The lumber needs to go from a moisture content of 40% to one of 10%. For the first kiln, the equation is $0.1 = 0.4\left(\frac{2}{3}\right)^w$. Because $0.4\left(\frac{2}{3}\right)^3 \approx 11.9\%$ and $0.4\left(\frac{2}{3}\right)^4 \approx 7.9\%$, the first kiln would produce this loss in 3 to 4 weeks.

For the second kiln, the equation is $0.1 = 0.4\left(\frac{3}{4}\right)^w$. Since $0.4\left(\frac{3}{4}\right)^3 \approx 12.7\%$ and $0.4\left(\frac{3}{4}\right)^5 \approx 9.5\%$, the second kiln would produce this loss in 4 to 5 weeks.

Amount of Moisture

Week	Fraction of Moisture Removed	Total Fraction of Moisture Removed	Fraction of Moisture Remaining
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$	$\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$	$\frac{4}{9}$
3	$\frac{1}{3} \times \frac{4}{9} = \frac{4}{27}$	$\frac{5}{9} + \frac{4}{27} = \frac{19}{27}$	$\frac{8}{27}$
4	$\frac{1}{3} \times \frac{8}{27} = \frac{8}{81}$	$\frac{19}{27} + \frac{8}{81} = \frac{65}{81}$	$\frac{16}{81}$
5	$\frac{1}{3} \times \frac{16}{81} = \frac{16}{243}$	$\frac{65}{81} + \frac{16}{243} = \frac{211}{243}$	$\frac{32}{243}$