Applications

1. a.

Cutting Processes	Number of Ballots
1	3
2	9
3	27
4	81
5	243

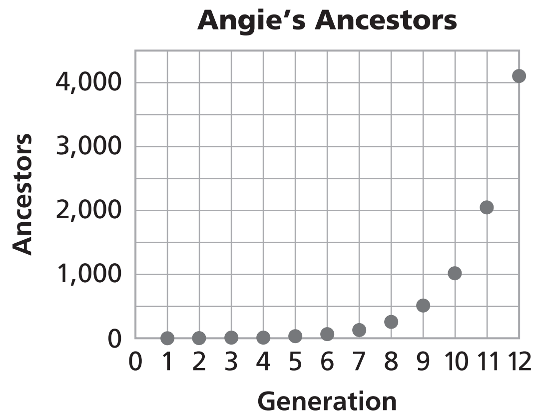
b. $3^{10} = 59{,}049$; 3^n

c. 13. After 13 cuts, there would be $1{,}594{,}323 = 3^{13}$ ballots, which is over 1 million ballots; but 3^{12} is less than 1 million.

2. Gabriel's conjecture is correct. Because each person is cutting ballots, each person would have 8 ballots, or 32 total. The reason why Chen's conjecture is incorrect is because each person is starting with their own piece of paper. His answer would be correct if after he cut his paper in half three times he handed his pile to the next person and they continued that way.

3. a. **Angie's Ancestors**

Generation	Number of Ancestors
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1,024
11	2,048
12	4,096



b. $a = 2^n$, where a is the number of ancestors and n is the generation number.

c. 8,190. You can find this by adding $2 + 4 + 8 + \ldots + 4{,}096$

Note: See the Math Background for a description of how to use a calculator to find the sum of a sequence. The ancestor pattern is identical to the pattern in the paper-cutting activity of Problem 1.1 and is very similar to the chessboard pattern in Problem 1.2. The only noticeable difference is that, in Problem 1.2, $n = 0$ doesn't make sense because there is not a square 0. (The graph for the rubas problem in Problem 1.2 is a translation of the graph that applies to both Problem 1.1 and the ancestor problem.) If you go back 40 generations, the number of ancestors exceeds the number of people who have ever lived!

4. a. square 33: $2^{32} = 4{,}294{,}967{,}296$

square 34: $2^{33} = 8{,}589{,}934{,}592$

square 35: $2^{34} = 17{,}179{,}869{,}184$

b. square 41; this is because it is the number of rubas on square 32 times nine more factors of 2.

c. The display probably reads 1.09951162778 E12. There is an E in the middle of the number.

d. $1.09951162778 \times 10^{12}$ (There are occasions when the calculator display will not give the last few digits exactly.)

e. $2^{10} = 1.024 \times 10^3$

$2^{20} = 1.048576 \times 10^6$

$2^{30} = 1.073741824 \times 10^9$

$2^{40} = 1.09951162778 \times 10^{12}$

$2^{50} = 1.12589990684 \times 10^{15}$

f. Possible answer: To write a number in scientific notation, place a decimal in the original number to get a number greater than or equal to 1 but less than 10. To compensate for placing the decimal in the original number, multiply the new number by a power of ten that will give you back your original number.

5. 1.0×10^8

6. $2.9678900522 \times 10^{10}$

7. 1.19505×10^{13}

8. 643,999,001

9. 889,234

10. 34,348,567,000

Values given for Exercises 11–15 are for the standard screen of the TI-83. Different calculators will give different results.

11. a. 20

b. 20

c. 9

d. 4

12. a. 10

b. 5

c. 8; **Note:** The question asks about $1{,}000(n^n)$ and not $(1{,}000n)^n$. Students often get confused by the notation $1{,}000n^n$, which means $1{,}000(n^n)$.

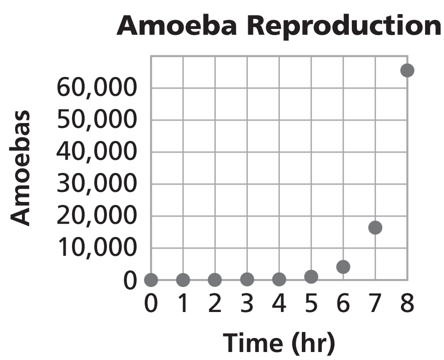
13. a. Amoeba Reproduction

Time (hr)	Amoebas
0	1
1	4
2	16
3	64
4	256
5	1,024
6	4,096
7	16,384
8	65,536

b. $a = 4^t$; time t is the independent variable and the number of amoebas a is the dependent variable.

c. 10 hours; at 10 hours, there will be 1,048,576 amoebas present

d.



e. The pattern of change in the number of amoebas is similar to the pattern of change in the number of ancestors in ACE #2 because it is exponential. The difference between the two patterns is that the number of amoebas increases more rapidly than the number of ancestors because the number of amoebas quadruples at each stage, while the number of ancestors only doubles. This doubling pattern is also similar to the pattern of cutting a piece of paper to make ballots and to the rubas problem.

14. a. Plan 1

Day	1	2	3	4	5	6	7	8	9	10	11	12
Donation	$1	$2	$4	$8	$16	$32	$64	$128	$256	$512	$1,024	$2,048

Plan 2

Day	1	2	3	4	5	6	7	8	9	10
Donation	$1	$3	$9	$27	$81	$243	$729	$2,187	$6,561	$19,683

Plan 3

Day	1	2	3	4	5	6	7	8
Donation	$1	$4	$16	$64	$256	$1,024	$4,096	$16,384

b. Plan 1: $d = 2^n$

Plan 2: $d = 3^n$

Plan 3: $d = 4^n$

c. All three plans are exponential functions because each can be written with a growth factor.

d. Plan 2

15. a. Grandmother's and Aunt Josie's plans are linear and Uncle Sebastián's and Father's plans are exponential.

b. Grandmother: $a = 2n - 1$

Father: $a = 3^{n-1}$

Aunt Josie: $a = 1.5n + 0.5$

Uncle Sebastián: $a = 2^{n-1}$

n is the independent variable and a is the dependent variable in all four equations.

c. Grandmother: $39

Father: $1,162,261,467

Aunt Josie: $30.50

Uncle Sebastián: $524,288

16. a. Graph 1 represents $y = 2^x$ because it is a curve. Graph 2 represents $y = 2x + 1$ because it is a straight line.

b. In both graphs, the horizontal change is constant. In the graph of $y = 2x + 1$, the vertical change is also constant. In the graph of $y = 2x$, the vertical change increases.

c. Graph 1 represents an exponential function because its equation is $y = 2^x$. Graph 2 represents a linear equation because its equation is $y = 2x + 1$.

17. a. linear

b. $y = 2.5x + 10$

18. a. exponential

b. $y = 6^x$

19. neither

20. a. exponential

b. $y = 2^{x+1}$

21. neither (In fact, this is quadratic. Students will study quadratic relationships in the unit *Frogs, Fleas, and Painted Cubes*).

CONNECTIONS

22. 2^4

23. 10^7

24. $(2.5)^5$

25. 1,024

26. 100

27. 19,683

28. Because 5^2 means $5 \cdot 5$ and 5^4 means $5 \cdot 5 \cdot 5 \cdot 5$, 5^4 also equals $5^2 \cdot 5^2 = 25 \cdot 25 = 625$.

29. Because 5^{11} has one more factor of 5 than 5^{10} has, it equals $5^{10} \cdot 5 = 9{,}765{,}625 \cdot 5 = 48{,}828{,}125$.

30. A

31. 10^9

32. 9^6 is less than 1 million. Possible explanation: The product of six 9s must be less than the product of six 10s, which is 10^6 or 1 million.

33. 3^{10} is less than 1 million. Possible explanation:

$3^{10} = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5$, which is less than 10^6.

34. 11^6 is greater than 10^6 or 1 million, because to find 11^6, you multiply 11 by itself six times. The result must be greater than if you multiply 10 by itself 6 times.

35. $5^3 = 125$

36. 2^6 or 4^3

37. 3^4

38. 5^5

39. 2^{10} or 4^5

40. 2^{12} or 4^6

41. **a.** After 20 cuts; $1{,}048{,}576 \div 250 \approx 4{,}194$ in.; about 349.5 ft; after 30 cuts; $1{,}073{,}741{,}824 \div 250 \approx 4{,}294{,}967$ in.; about 67.8 mi.

b. 12 cuts; a foot-high stack has $250 \times 12 = 3{,}000$ ballots. Because 11cuts gives 2,048 ballots and 12 cuts gives 4,096 ballots, Chen would have to make at least 12 cuts to get 3,000 ballots.

c. The distance to the moon, in inches, is approximately

$$240{,}000 \times 5280 \times 12 \approx 1.52 \times 10^{10}.$$

This means we need about $1.52 \times 10^{10} \times 250 \approx 3.80 \times 10^{12}$ sheets. To obtain this many sheets, we need 42 cuts, since $2^{42} \approx 4.39 \times 10^{12}$ and 41 cuts gives only half this number.

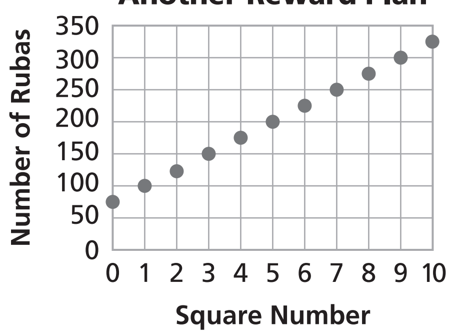
42. square 10: \$5.12; square 20: \$5,242.88; square 30: \$5,368,709.12; square 40: \$5,497,558,138.88; square 50; about $\$5.63 \times 10^{12}$; square 60: about $\$5.76 \times 10^{15}$.

43. When $n = 64$, the number of rubas is $2^{64-1} \approx 9.22 \times 10^{18}$. The stack would have been about $(9.22 \times 10^{18}) \times 0.06 = 5.53 \times 10^{17}$ in., or 4.61×10^{16} ft, or 8.73×10^{12} miles high.

44. **a.** $r = 100 + 25(n - 1)$ or $r = 25n + 75$

b. The graph will look linear. You might want to ask students to draw the graph.

Another Reward Plan



c. 2,125 rubas; 6,750 rubas

45. slope: 3; y-intercept: -10

46. slope: -5.6; y-intercept: 1.5

47. slope: $\frac{2}{5}$; y-intercept: 15

48. any equation with a coefficient of less than $\frac{2}{5}$; Possible answers: $y = \frac{1}{5}x + 3$; $y = \frac{1}{6}x + 15$

Extensions

49. a. Equation 1: $r = 3^2 - 1 = 9 - 1 = 8$

Equation 2: $r = 3^{2-1} = 3^1 = 3$

b. Equation 1: $r = 3^{10} - 1 = 59{,}048$

Equation 2: $r = 3^{10-1} = 3^9 = 19{,}683$

c. The equations give different values of r because subtraction is used differently. In one equation, 1 is subtracted from n and the result becomes the exponent of 3; in the other, n is used as the exponent of 3, and 1 is subtracted from the result.

Note: For $n \geq 0$, the result of Equation 1 will always be greater than that of Equation 2 because the exponent is greater. Subtracting from greater exponential contribution is almost insignificant.

d. The equation $r = 3^{n-1}$ represents an exponential function whose growth rate is 3. That is, as n increases by 1, r increases by a factor of 3. The equation $r = 3^n - 1$ does not represent an exponential function, because as n increases by 1, r does not increase by a constant factor.

50. a. $b = 2^n$, where b is the number of ballots made and n is the number of cuts.

b. From the table, 2^0 must equal 1. When you evaluate 2^0 with a calculator, the answer is 1.

c. The value of any non-zero number b raised to a power of 0 is 1.

Note: Talk with students about why this makes sense. Because $b^1 = b$ and because exponents tell us how many factors of the base to use, $b \times b^0 = b^{1+0}$ must equal to 1 + 0 factors of b, which is just b; so b^0should be 1 to make all of the other ideas about exponents work out. Some students might say that for the pattern to continue backwards, 2^0 must equal 1. Explain that mathematicians decided the world of mathematics would make more sense if b^0 is defined as 1 for $b \neq 0$.

51. a.

Reward Plan 1

Square	Number of Rubas on Square	Total number of Rubas
1	1	1
2	2	3
3	4	7
4	8	15
5	16	31
6	32	63
7	64	127
8	128	255
9	256	511
10	512	1,023

b. Each entry in the total column is 1 less than the entry in the next individual square column. Another pattern is double the total rubas at a square and add 1 to get the total rubas at the next square. The relationship in the second column represents an exponential function because as the square number increases by 1, the number of rubas increases by a factor of 2. The third or last column is not an exponential function, because as the square number increases by 1, the number of rubas does not increase by a constant factor.

c. $t = 2^n - 1$

d. The total t will exceed 1,000,000 when 20 squares have been covered;

$t = 2^{20} - 1 = 1{,}048{,}575$ rubas.

e. With all 64 squares covered, the total would be $t = 2^{64} - 1 \approx 1.84 \times 10^{19}$ rubas.

52. a. Possible answer: The King would most likely pick Plan 4, since that requires him to pay the least amount of money.

b. Possible answer: The peasant would most likely pick Plan 1, since that would earn her the most money.

c. Check students' work.