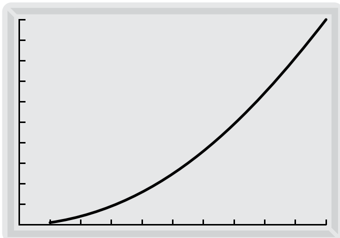
Applications

1. a. 25, 36

b. $s = n^2$

c. The numbers seem to be increasing by a greater amount each time. The square number increases by consecutive odd integers: 3, 5, 7, 9, . . .



X	Y1
0	0
1	1
2	4
3	9
4	16
5	25
6	36

X=0

X	Y1
6	36
7	49
8	64
9	81
10	100
11	121
12	144

X=12

2. a. 2, 6, 12, 20

b. 30, 42

c. The rectangular number increases by consecutive even integers: 4, 6, 8, 10, . . .

d. 56, 72; the 7th number is 14 greater than the 6th number, and the 8th number is 16 greater than the 7th number.

e. $r = n(n + 1)$, where r is the rectangular number.

3. a. $\frac{18(18 + 1)}{2} = \frac{18(19)}{2} = 171$

b. Yes; $\frac{20(20 + 1)}{2} = \frac{20(21)}{2} = 210$

Note: Students may also continue the table to answer this.

4. a. Sam is correct because if you look at the triangles in Problem 3.1, each row of a triangle represents a counting number: 1, 2, 3. . . . Therefore, the equation for triangular numbers, $\frac{n(n + 1)}{2}$ represents the sum of the numbers 1 to n.

b. 55

c. 120

d. $\frac{n(n + 1)}{2}$

5. Rectangular; it satisfies the equation $r = n(n + 1)$ where the 10th rectangular number $10(11) = 110$.

6. Triangular; it satisfies the equation $t = \frac{n(n + 1)}{2}$ where the 11th triangular number: $66 = \frac{11(12)}{2}$.

7. Square; it satisfies the equation $s = n^2$, where the 11th square number is $121 = 11^2$.

8. None; it does not satisfy any of the associated equations.

9. a. The eight people on each side shake hands with 8 others, so $8^2 = 64$ handshakes will be exchanged.

b. 28; $7 + 6 + 5 + 4 + 3 + 2 + 1 = \frac{7 \times 8}{2} =$ 28, or each of 8 people shake hands with the other 7, but as this counts each handshake twice, 28 handshakes will be exchanged.

c. 12; $6 \times 2 = 12$

10. a. 15; $5 \times 3 = 15$

b. 10; $4 + 3 + 2 + 1 = \frac{4 \times 5}{2} = 10$, or each of the 5 people high five with the other 4, but as this counts each high five twice, 10 high fives will be exchanged.

11. a. 10; $4 + 3 + 2 + 1 = \frac{4 \times 5}{2} = 10$, or each of 5 rooms connects with each other 4 by cables, but as this counts two cables between each two rooms, 10 cables will be needed.

b. 21; $6 + 5 + 4 + 3 + 2 + 1 = \frac{6 \times 7}{2} = 21$.

c. They are the same situation mathematically, where the cables are associated with high fives and rooms are associated with people.

12. Possible answers: P represents the area of a rectangle with dimensions n units by $n - 1$ units. P could also represent the number of handshakes between a team of n players and another team of $n - 1$ players.

13. Possible answer: P represents the area of a rectangle with dimensions 2 units by n units.

14. Possible answer: P represents the area of a rectangle with dimensions n units by $n - 2$ units.

15. Possible answer: P represents the area of a rectangle with a perimeter of 32 units.

16. a. Graph II; Possible explanation: The equation $h = \frac{n(n-1)}{2}$, which represents the relationship between the number of high fives and the number of team players, tells us that the graph will have x-intercepts of 0 and 1.

b. Graph III; Possible explanation: The area of rectangles with a fixed perimeter increases and then decreases as the length of a side increases.

c. Graph I; Possible explanation: Since the equation for the relationship described is $y = (x + 2)(x - 3)$, the x-intercepts must be -2 and 3.

d. Graph IV; Possible explanation: The nth triangular number can be represented by the equation $T = \frac{n(n+1)}{2}$. This equation that tells us the graph has two x-intercepts, 0 and -1.

17.

x	y
0	0
1	1
2	3
3	6
4	10
5	15
6	21

18.

x	y
0	0
1	3
2	8
3	15
4	24
5	35
6	48

19.

x	y
0	0
1	4
2	6
3	6
4	4
5	0
6	−6

20. This is not a quadratic function. **Note:** If the point $(5, -18)$ were $(5, -20)$, this would be a quadratic function.

21. This is a quadratic function with a minimum point.

22. This is a quadratic function with a minimum point.

23. This is not a quadratic function. **Note:** This has symmetry about the line $x = 0$, but this has two linear segments; its equation is $y = |x| + 1$.

24. This is a quadratic function with a minimum point.

25. a. In each equation, second differences are constant, which means that all the equations are quadratic. The constant second differences for each equation are equal to $2a$, where a is the coefficient of x^2.

$y = 2x^2$

x	y
0	0
1	2
2	8
3	18
4	32
5	50

First Differences	Second Differences
2	
6	4
10	4
14	4
18	4

$y = 3x^2$

x	y
0	0
1	3
2	12
3	27
4	48
5	75

First Differences	Second Differences
3	
9	6
15	6
21	6
27	6

$y = \frac{1}{2}x^2$

x	y
0	0
1	0.5
2	2
3	4.5
4	8
5	12.5

First Differences	Second Differences
0.5	
1.5	1
2.5	1
3.5	1
4.5	1

$y = -2x^2$

x	y
0	0
1	–2
2	–8
3	–18
4	–32
5	–50

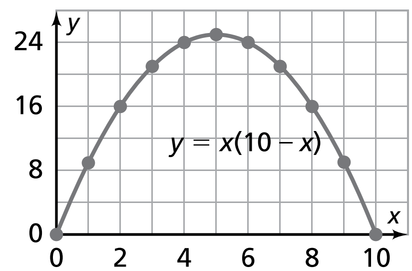
First Differences	Second Differences
–2	
–6	–4
–10	–4
–14	–4
–18	–4

b. Since these are quadratic equations, second differences will be constant and will be equal to twice the number multiplied by x^2. For $y = -4x^2$, second differences will be -8. For $y = \frac{1}{4}x^2$, second differences will be $\frac{1}{2}$. For $y = ax^2$, second differences will be $2a$.

26. a. Table of (x, y) Values

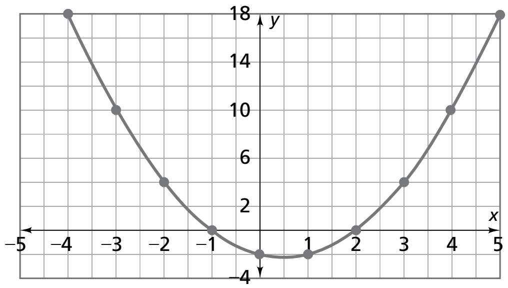
x	y
0	0
1	9
2	16
3	21
4	24
5	25

b. $y = x(10 - x)$



Because the maximum point is given, I can find the line of symmetry and complete the graph by plotting the corresponding point on the right side for each point on the left side.

27.



28. If you extend the table, you will get the following values: (−1, 15), (−2, 24), (−3, 35), (−4, 48), (−5, 63). The second difference is 2.

Connections

29. a.

4	$4x$	8
x	x^2	$2x$
	x	2

The expanded form is $x^2 + 6x + 8$ and the factored form is $(x + 2)(x + 4)$.

1	$1x$	8
x	x^2	$8x$
	x	8

The expanded form is $x^2 + 9x + 8$ and the factored form is $(x + 8)(x + 1)$. **Note:** The sides of area models cannot have negative values, so factored forms like $(x - 2)(x - 4)$ aren't possible even though they give you the terms 8 and x^2.

b. No; the expanded form is $x^2 + 6x + 5$ and the factored form is $(x + 1)(x + 5)$.

30. $2x^2 + 9x + 9$ or $(2x + 3)(x + 3)$

31. a. $2x^2 + 7x + 6$

	x	2
$2x$	$2x^2$	$4x$
3	$3x$	6

$x^2 + 6x + 8$

	x	4
x	x^2	$4x$
2	$2x$	8

$x^2 + 6x + 8$ is easier to do because x^2 has a coefficient of one.

b. $2x^2 + 7x + 6 = (2x + 3)(x + 2)$;
$x^2 + 6x + 8 = (x + 4)(x + 2)$

32. $5x - x^2$

33. $x^2 + 4x + 3$

34. $x^2 + 2x - 3$

35. $3x^2 + 15x$

36. $2x^2 + 7x + 3$

37. $2x^2 + 5x - 3$

38. $(x - 8)(x - 1)$

39. $2x(2x - 3)$

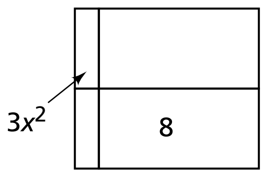
40. $(3x + 2)(x + 4)$

41. $2x(2x + 3)$

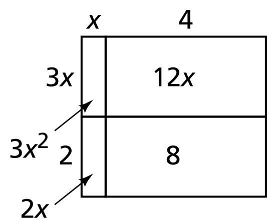
42. $(4x + 3)(x - 1)$

43. $x(x + 1)(x - 3)$

44. a. Subdivide a rectangle into four parts. Label the area of one of the smaller rectangles as $3x^2$ and the one diagonal to it as 8.



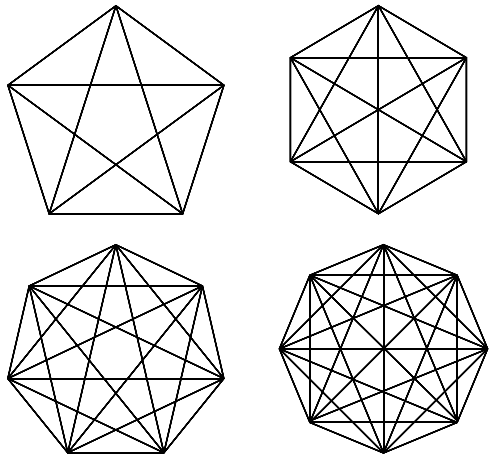
Use these areas to find the dimensions of these two smaller rectangles. The dimensions of the rectangle whose area is $3x^2$ are $3x$ and x and the dimensions of the rectangle with area 8 are 4 and 2. The sum of the areas the other two rectangles should equal $14x$, so they are $2x$ and $12x$.



The dimensions of the original rectangle are $3x + 2$ and $x + 4$.
Note: There are several ways to do this.

b. Look at the factors of the coefficient of x^2 and the factors of the constant term. Put these values in the factor pairs: $(\underline{?}x + \underline{?})(\underline{?}x + \underline{?})$. Use the Distributive Property to check if the coefficient of x is correct.

45. a. A pentagon has 5 diagonals, a hexagon has 9 diagonals, a heptagon has 14 diagonals and an octagon has 20 diagonals.

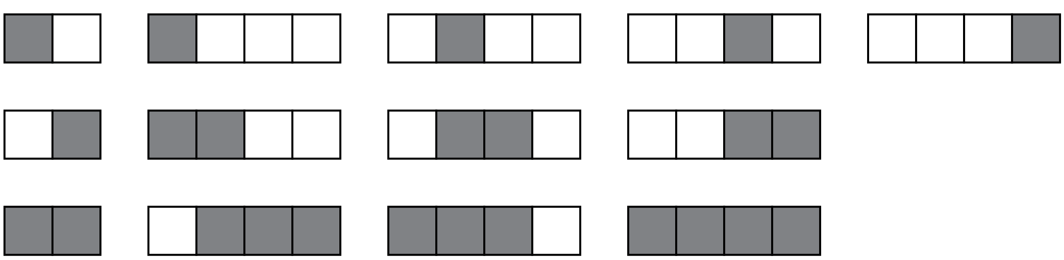


b. An n-sided polygon has $\frac{n(n-3)}{2}$ diagonals. This problem could be solved like the high fives problem. Each of n points exchanges high fives, or shares a diagonal, with the other $n - 3$ points (excluding the adjacent points and itself). Since this counts each high five twice, the expression is divided by 2.

46. a. The first train has 1 rectangle, the square itself. The second train has 3 rectangles, as shown below. The third train has 6 rectangles, as in the problem. The fourth train has 10 rectangles, as shown below. The fifth train has 15 rectangles (5 one-square rectangles, 4 two-squares rectangles, 3 three-square rectangles, 2 four-square rectangles and 1 five-square rectangle.) (See Figure 1.)

Figure 1

2nd train **4th train**



b.

Number of Rectangles in the First Ten Trains

Train	Number of Rectangles
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55

c. The number of rectangles increases by a greater amount each time. The pattern of increase is 2, 3, 4, 5, 6,. . . So, we could expect an increase of 15 from the 14th train to the 15th train. The table below shows that there are 120 rectangles in the 15th train.

Number of Rectangles in Trains 11 Through 15

Train	Number of Rectangles
11	66
12	78
13	91
14	105
15	120

d. $r = \frac{n(n+1)}{2}$ where r is the number of rectangles. **Note:** The numbers are the same as the triangular numbers, an observation that students may use to solve this equation.

e. $r = \frac{15(15+1)}{2} = \frac{15(16)}{2} = 120$ rectangles

47. **a.** ≈ 78.5 cm^2

b. ≈ 78.5 centimeter cubes

c. 10 layers

d. ≈ 785 cm^3; this is the product of the number of cubes in one layer and the number of layers that fill the can.

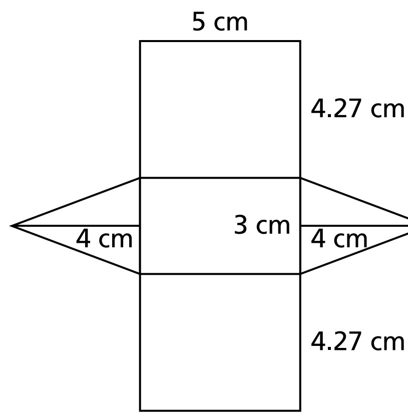
e. ≈ 31.4 cm

f. ≈ 314 cm^2

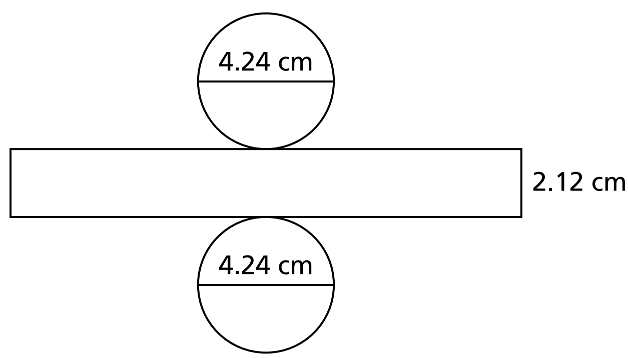
g. ≈ 471 cm^2; this is the sum of twice the area of the base and the area of the paper label.

48 **a.**

Box A



Box B



b. The surface area of Box A = 69.7 cm^2, and the surface area of Box B = 56.48 cm^2. To find the surface area of Box A, add the area of the base, $3 \times 5 = 15$ cm^2, the total area of the two triangles, $2(1.5 \times 4) = 12$ cm^2, and the total area for the two side rectangles, $2(5 \times 4.27) = 42.7$. (By the Pythagorean Theorem, the hypotenuse of the triangle and the other dimension of the side rectangles is $\sqrt{1.5^2 + 4^2} \approx 4.27$.) So, the surface area is $15 + 12 + 42.7 = 69.7$ cm^2. To find the surface area of Box B, add the total area of the two circles, $2 \times \pi(2.12)^2 \approx 28.24$ cm^2, to the area of the rectangle, $2.12 \times (2 \times \pi \times 2.12) \approx 28.24$ cm^2. So the surface area is $28.24 + 28.24 = 56.48$ cm^2.

c. Box A will require more cardboard to construct, since it has a greater surface area.

49. None of the above.

50. Quadratic

51. Exponential

52. None of the above

53. D

54. H; by looking at the coefficient of x^2, H is the only parabola that can have a minimum point because the coefficient is positive. The other three parabolas open down, because their coefficients are negative.

55. **a.** $-6x^2 + 3x$

b. $9x - 3x^2$

Extensions

56. **a.** $(1 + 100) + (2 + 99) + \dots (99 + 2) + (100 + 1) = \frac{100(101)}{2} = 5{,}050$.

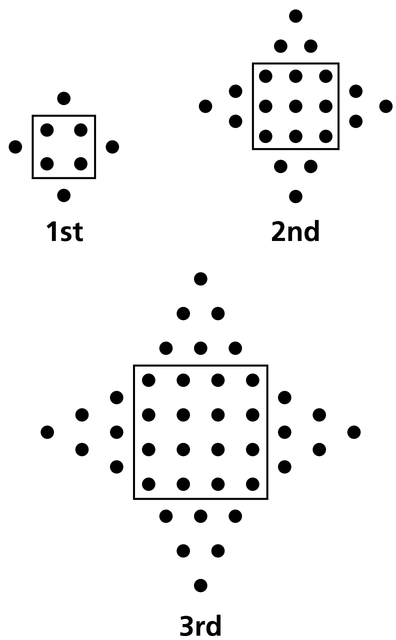
b. This idea could be represented by the equation $s = \frac{n(n + 1)}{2}$, where s is the sum of the first s whole numbers.

c. This method is the same as Gauss's method in the Did You Know? It pairs the numbers in a number sentence, which doubles the value. Then it divides by 2 to get the sum.

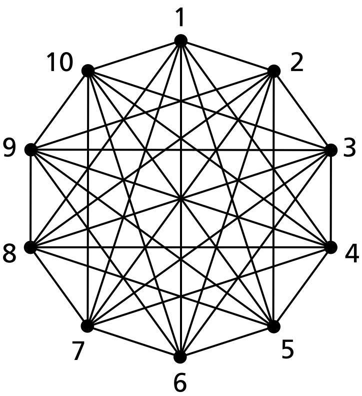
57. **a.** 8, 21, 40

b. 65, 96, 133

c. $s = (n + 1)^2 + \frac{4n(n + 1)}{2}$ **Note:** Students may not be able to find this equation. You might help them to write an equation by pointing out that each star has a center square and four triangular points. Each star number n is thus composed of 4 times the nth triangular number plus the $(n + 1)$th square number. For example, the first star number has a center of 4 (the second square number) and four "points" of 1 (the first triangular number); the second star number has a center of 9 (the third square number) and four points of 3 (the second triangular number).



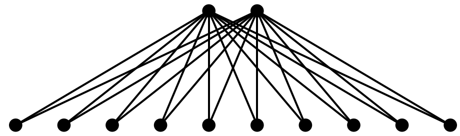
58. a. 45; $\frac{10(9)}{2} = 45$, or each of the 10 classmates shakes hands with 9 others, but as this counts each handshake twice, 45 handshakes occur. This answer can also be expressed in different types of diagrams.

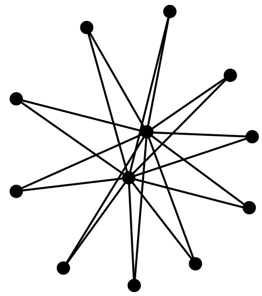


	1	2	3	4	5	6	7	8	9	10
1		•	•	•	•	•	•	•	•	•
2			•	•	•	•	•	•	•	•
3				•	•	•	•	•	•	•
4					•	•	•	•	•	•
5						•	•	•	•	•
6							•	•	•	•
7								•	•	•
8									•	•
9										•

People: A, B, C, D, E, F, G, H, I, J
Handshakes: AB, AC, AD, AE, AF, AG, AH, AI, AJ
BC, BD, BE, BF, BG, BH, BI, BJ
CD, CE, CF, CG, CH, CI, CJ
DE, DF, DG, DH, DI, DJ
EF, EG, EH, EI, EJ
FG, FH, FI, FJ
GH, GI, GJ
HI, HJ
IJ

b. Each of the 2 classmates shakes hands with 11 others, but as one handshake is counted twice (when they shake hands with each other), there are $2(11) - 1 = 21$ handshakes in all. This answer can also be expressed in different types of diagrams.



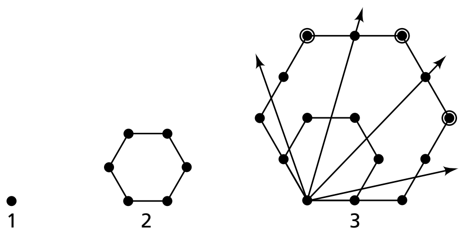


	1	2	3	4	5	6	7	8	9	10	11
11	•	•	•	•	•	•	•	•	•	•	
12	•	•	•	•	•	•	•	•	•	•	•

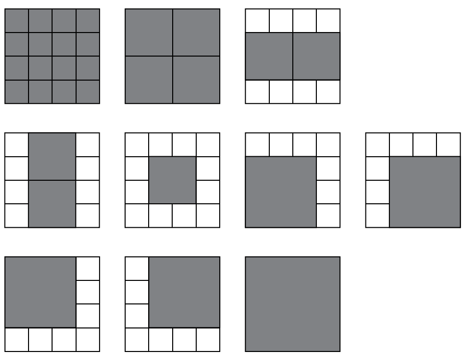
59. a. 1, 6, 15

b. 28, 45

c. $h = n(2n - 1)$. **Note:** Students may not be able to find this equation. You might help them write an equation by pointing out that there are four lines along which they can add $1 + 2 + 3 + \ldots + n$ dots. The equation for the triangular numbers, $\frac{n(n+1)}{2}$, is multiplied by 4 to get this sum. However, this counts the dots circled on the diagram an extra 3 times for each hexagon, a total of $3n$ times for the whole figure. Thus, the complete equation for hexagonal numbers is $h = \frac{4n(n+1)}{2} - 3n = 2n^2 + 2n - 3n = 2n^2 - n = n(2n - 1)$.

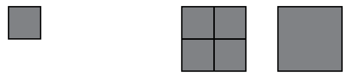


60. a. Of the 14 remaining squares, 9 are 2-by-2 squares, 4 are 3-by-3 squares, and 1 is a 4-by-4 square.

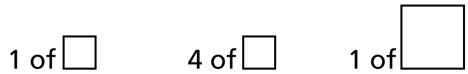


b. Possible answer: The squares found in a 1-by-1 grid, a 2-by-2 grid and a 3-by-3 grid are shown below. Each grid contains n^2 more squares than the previous grid, so an equation for the number of squares in an n-by-n grid is $s = n^2 + (n - 1)^2 + \ldots + 1^2$, where s is the number of squares.

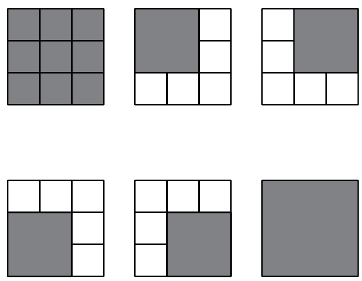
1 by 1 grid **2 by 2 grid**



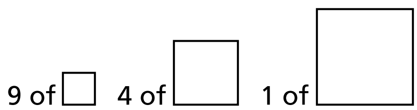
1 square: **5 squares:**



3 by 3 grid



14 squares:



This shows that the pattern is a sum of squares.

61. a. The graph of $y_1 = x + 1$ is a straight line with slope 1 and y-intercept (0, 1). The graph of $y_2 = (x + 1)(x + 2)$ is a parabola with a minimum point at $(-1.5, -0.25)$ and x-intercepts at $(-1, 0)$ and $(-2, 0)$. The graph of $y_3 = (x + 1)(x + 2)(x + 3)$ increases as x increases, then decreases, then increases again. It has three x-intercepts at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$. The graph of $y_4 = (x + 1)(x + 2)(x + 3)(x + 4)$ is shaped like the letter W. It has two local minimum points, a local maximum point, and four x-intercepts at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$, $(-4, 0)$.
Note: The terms *local minimum* and *local maximum* will be introduced in future mathematics courses. They refer to minimums and maximums over a given part of the graph, which are not necessarily the minimum or maximum for the entire graph.

b. The equation $y_1 = (x + 1)$ has constant first differences. The equation $y_2 = (x + 1)(x + 2)$ has constant second differences. The equation $y_3 = (x + 1)(x + 2)(x + 3)$ has constant third differences. The equation $y_4 = (x + 1)(x + 2)(x + 3)(x + 4)$ has constant fourth differences.