Answers Applications Connections Extensions

Investigation 4

ACE Assignment Choices

Differentiated Instruction
Solutions for All Learners

Problem 4.1

Core 1–4

Other *Connections* 31, 32; *Extensions* 51–53; unassigned choices from previous problems

Problem 4.2

Core 5–8, 11–17

Other *Applications* 9, 10; *Connections* 33–35; *Extensions* 54; unassigned choices from previous problems

Problem 4.3

Core 18–22, 25, 26

Other *Applications* 23, 24; *Connections* 36–40; *Extensions* 55; unassigned choices from previous problems

Problem 4.4

Core 27–30

Other *Connections* 41–50; *Extensions* 56, 57; unassigned choices from previous problems

Adapted For suggestions about adapting ACE exercises, see the CMP *Special Needs Handbook*.

Connecting to Prior Units 33: *Covering and Surrounding, Stretching and Shrinking*; 34, 35: *Filling and Wrapping*; 33: *Covering and Surrounding*; 34, 35: *Filling and Wrapping*; 41, 43–45: *Covering and Surrounding*

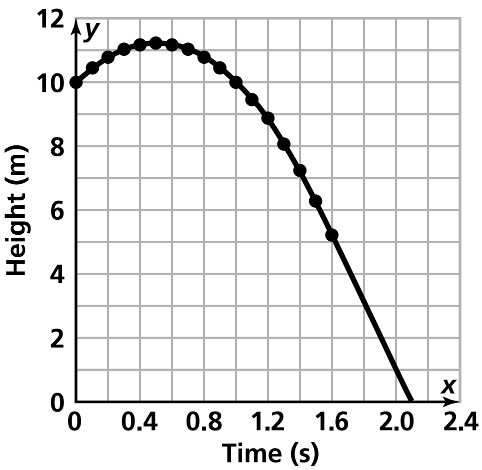
Applications

1. a. At 5 seconds, the flare will have traveled to a maximum height of 400 ft.
   b. The flare will hit the water when the height is 0 ft, which will occur at 10 s.
   c. In a graph, the maximum point represents the maximum height of the flare, and the right-hand x-intercept represents the point at which the flare hits the water. In a table, the entry for when the height is its greatest represents the maximum height reached by the flare, and the entry for when the height is once again 0 represents the point at which the flare hits the water.
2. a. The rocket will travel to a height of 148 feet. It reaches this maximum height after 2 seconds.
   b. The rocket was launched at a height of 84 feet above ground level.
   c. It will take 4 seconds for the rocket to return to the height from which it was launched.
3. a. The ball is released at about 6.5 ft (the y-intercept).
   b. The ball reaches its maximum height, about 17.5 ft, at about 0.8 seconds.
   c. The ball would reach the basket just after 1.5 seconds.
4. a. **Height of a Diver After t Seconds**

Time (t)	Height (h)
0	10
0.1	10.441
0.2	10.784
0.3	11.029
0.4	11.176
0.5	11.225
0.6	11.176
0.7	11.029
0.8	10.784
0.9	10.441
1.0	10
1.1	9.461
1.2	8.824
1.3	8.089
1.4	7.256
1.5	6.325

ACE ANSWERS 4

b. Diving From the Platform



c. The diver hits the water's surface when the height is 0, which happens at between 2 and 2.1 seconds. In the graph, this is the x-intercept. In the table, it is the entry for when height is 0.

d. The diver will be 5 m above the water's surface between 1.6 and 1.7 seconds.

e. The diver is falling at the greatest rate just before hitting the water's surface. In the table, this is when the difference between successive height values is the greatest. In the graph, this is where the curve has the steepest downward slope.

5. a. The maximum height is about 15.06 ft, which occurs after about 0.56 seconds. (Note: Students can find this by making a table or a graph of the equations.)

b. Her feet hit the water when the height is 0, which occurs at about 1.53 seconds.

c. The board is 10 ft above the water's surface.

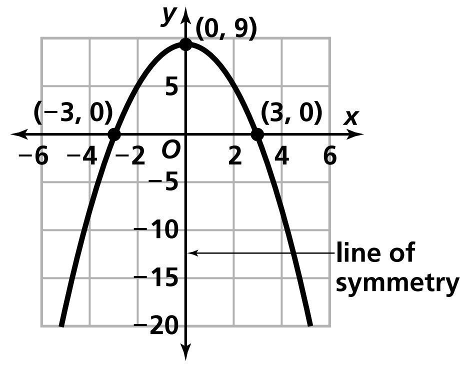
6. a. The maximum height is 44 ft, which is reached at 1.5 seconds. You could find this in a table of time versus height by locating the maximum height. You could find this in a graph by determining the height at the maximum point of the parabola.

b. The ball hits the ground just after 3.1 seconds. You could find this in a table of time versus height by locating the value for time when height is 0. You could find this in a graph by determining the time at the point at which the parabola crosses the x-axis.

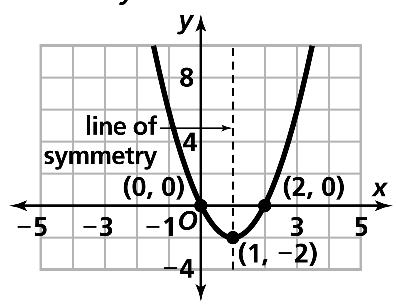
c. The ball begins rising rapidly and then slows its ascent until it reaches the maximum height of 44 ft. It then starts to fall, slowly at first and gaining speed on the way down until it hits the ground.

d. The ball is 8 ft above ground when thrown.

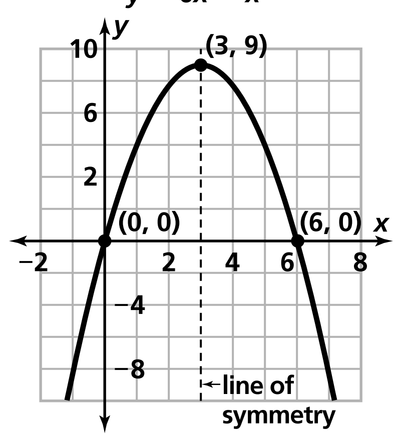
7. $y = 9 - x^2$



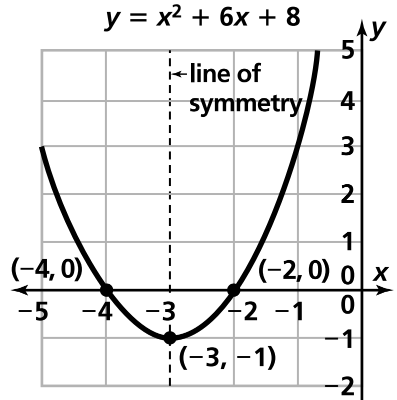
8. $y = 2x^2 - 4x$



9. $y = 6x - x^2$



10. $y = x^2 + 6x + 8$

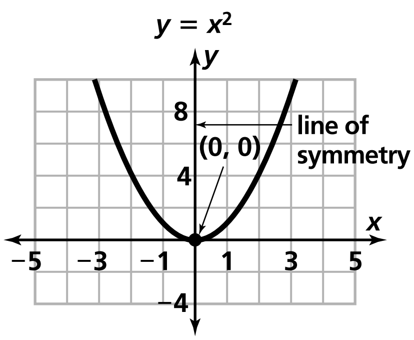


11. **a.** If the sign of the coefficient of the x^2 term is negative, the graph will have a maximum point. If it is positive, the graph will have a minimum point.

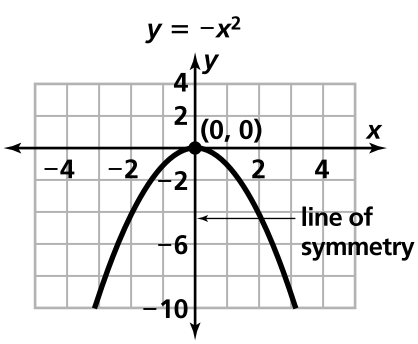
b. The x-intercepts are the values that make each factor in the factored form of the equation equal to 0. The y-intercept is the constant term in the expanded form of the equation.

c. If there are two x-intercepts, the distances from each x-intercept to the line of symmetry are the same. If there is only one intercept, it is on the line of symmetry. There is not any apparent relationship between the y-intercept and the line of symmetry.

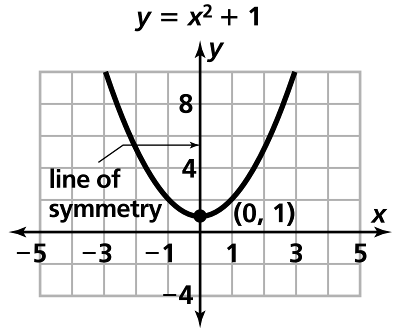
12. We can predict that this is a parabola with x-intercepts and minimum at $(0, 0)$.



13. We can predict that this is a parabola with x-intercepts and maximum at $(0, 0)$.

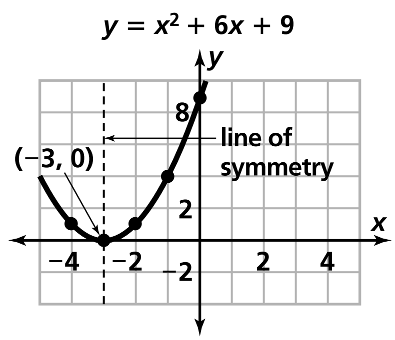


14. We can predict that this is a parabola with a minimum, and the y-intercept at $(0, 1)$.

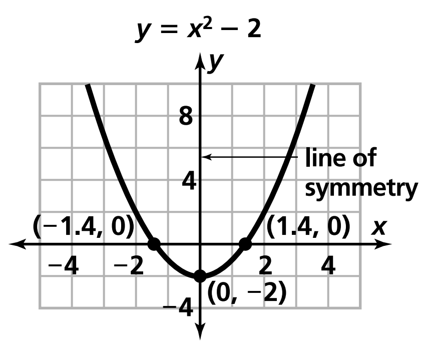


Note to the teacher: This graph does not have real roots; that is, it does not cross the x-axis. If $y = 0$, then $x^2 = -1$, so x is a complex number.

15. If we factor this we have $y = (x + 3)^2$. From this, we can predict this is a parabola with minimum and x-intercept at $(0, -3)$. We can predict the y-intercept from $y = x^2 + 6x + 9$; it is $(0, 9)$.

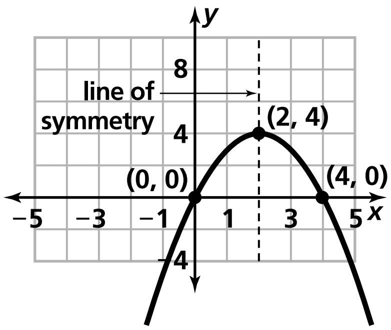


16. We can predict that this is a parabola with a minimum and y-intercept at $(0, -2)$.



17. We can predict that this is a parabola with x-intercepts at 0 and 4, and a vertex at (2, 4). From the expanded form $y = 4x - x^2$ we can predict there will be a maximum at (2, 4).

$y = x(4 - x)$



18. This is not a quadratic relationship. (Note: If the point (5, −18) were (5, −20), this would be a quadratic relationship.)

19. This is a quadratic relationship with a minimum point.

20. This is a quadratic relationship with a minimum point.

21. This is not a quadratic relationship. (Note: This has symmetry about the line $x = 0$, but this has two linear segments; its equation is $y = |x| + 1$)

22. This is a quadratic relationship with a minimum point.

23. a. In each equation, second differences are constant, which means that all the equations are quadratic. The constant second differences for each equation are equal to $2a$, where a is the coefficient of x^2. See tables below.

$y = 2x^2$

x	y
0	0
1	2
2	8
3	18
4	32
5	50

First Differences	Second Differences
2	
	4
6	
	4
10	
	4
14	
	4
18	

$y = 3x^2$

x	y
0	0
1	3
2	12
3	27
4	48
5	75

First Differences	Second Differences
3	
	6
9	
	6
15	
	6
21	
	6
27	

$y = \frac{1}{2}x^2$

x	y
0	0
1	0.5
2	2
3	4.5
4	8
5	12.5

First Differences	Second Differences
0.5	
	1
1.5	
	1
2.5	
	1
3.5	
	1
4.5	

$y = -2x^2$

x	y
0	0
1	−2
2	−8
3	−18
4	−32
5	−50

First Differences	Second Differences
−2	
	−4
−6	
	−4
−10	
	−4
−14	
	−4
−18	

b. Since these are quadratic equations, second differences will be constant and will be equal to twice the number multiplied by x^2. For $y = 5x^2$, second differences will be 10; for $y = ax^2$, second differences will be $2a$.

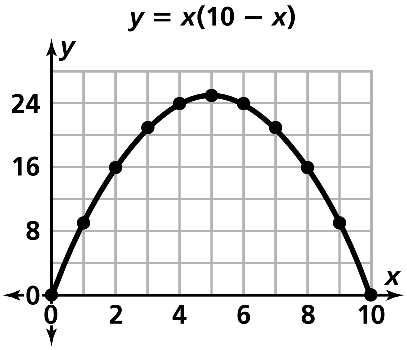
c. Yes, the equations are quadratic and the second difference for each is a constant.

24. a.

Table of (x, y) Values

x	y
0	0
1	9
2	16
3	21
4	24
5	25

b.



We know where the maximum point is, so we can find the line of symmetry and complete the graph by plotting a corresponding point on the right side for each point on the left side.

25. We know that the minimum point is where $x = 0.5$, so we can find the line of symmetry and complete the graph by plotting a corresponding point on the left side for each point on the right side. (Figure 7)

26. If you extend the table, you will get the following values: $(-1, 15)$, $(-2, 24)$, $(-3, 35)$, $(-4, 48)$, $(-5, 63)$. Note: The second difference is 2.

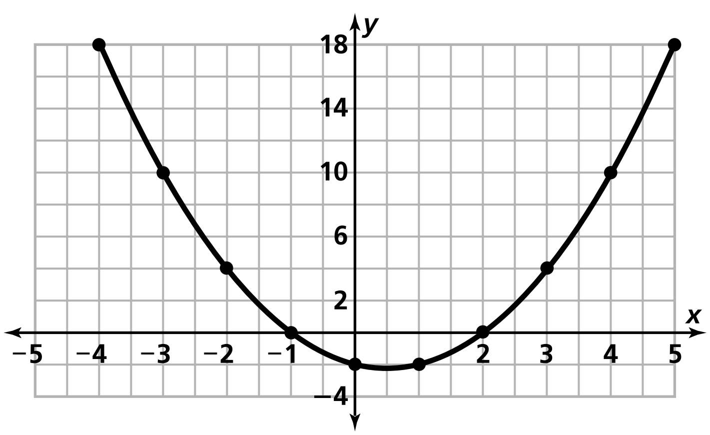
27. a. The 8 corners, or 8 cubes.

b. The cubes along the 12 edges that are not corner cubes, or $12 \times 10 = 120$ cubes.

c. The large cube has 6 faces, and each face contains $10 \times 10 = 100$ cubes with one face painted, a total of $6 \times 100 = 600$ cubes.

d. Removing the external cubes leaves $10 \times 10 \times 10 = 1{,}000$ unpainted cubes.

Figure 7



28. a. The unpainted cubes form a 10-by-10-by-10 cube on the inside of the large cube, which means the dimensions of the large cube must be 12-by-12-by-12, with 1,728 total cubes.

b. Each of the 6 faces on the cube contains $\frac{864}{6} = 144$ cubes with one face painted. There are 144 cubes arranged in a 12-by-12 square, which means the large cube must have the dimensions of 14-by-14-by-14, with 2,744 total cubes.

c. Each of the 12 edges contains $\frac{132}{12} = 11$ cubes painted on two faces, which means the large cube must have the dimensions of 13-by-13-by-13, with 2,197 total cubes.

d. Any cube would have 8 cubes painted on three faces, located at the 8 corners; we cannot tell the size of the large cube based on this information.

29. a. In the values for x, first differences are constant. In the values for x^2, second differences are constant. In the values for x^3, the third differences are constant.

x	x
1	1
2	2
3	3
4	4
5	5

x	x^2
1	1
2	4
3	9
4	16
5	25

x	x^3
1	1
2	8
3	27
4	64
5	125

b. In the table of value x, the pattern of change is similar to the pattern of the number of cubes with 3 or 2 faces painted because their first differences are constant. In the table of value x^2, the pattern of change is similar to the pattern of the number of cubes with 1 face painted because their second differences are constant. In the table of value x^3, the pattern of change is similar to the pattern

ACE ANSWERS 4

of the number of cubes with 0 faces painted because their third differences are constant.

30. $y_1 = 2(x - 1)$ is similar to the relationship of the number of cubes painted on two faces because they are both linear. $y_2 = (x - 1)^3$ is similar to the relationship of the number of cubes painted on 0 faces or total cubes because they are both cubic. $y_3 = 4(x - 1)^2$ is similar to the relationship for the number of cubes painted on one face because they are both quadratic. (Note: Students can observe the similarity from the form of equations or the pattern of changes in tables.)

Connections

31. a. Table 1: Each y-value is twice the previous y-value. The missing entry is (5, 800).

Table 2: Each y-value is 3 greater than the previous y-value. The missing entry is (2, 18).

Table 3: Each increase in the y-value is 2 greater than the previous increase. The missing entry is (7, 56).

Table 4: Each increase in the y-value is 2 less than the previous increase. The missing entry is (3, 16).

b. Table 1; $y_6 = 25(2^x)$

Table 2; $y_5 = 3(x + 4)$

Table 3; $y_2 = x(x + 1)$

Table 4; $y_3 = 25 - x^2$

c. Tables 3 and 4. In Tables 3 and 4, the second differences are constant.

d. Table 4. (0, 25).

e. The minimum is not visible in any of the tables, but if the tables are extended, there will be a minimum point.

32. a. The equations are equivalent. Possible explanation: When you graph the equations, the graphs are identical so the equations must be the same, or use the Distributive Property.

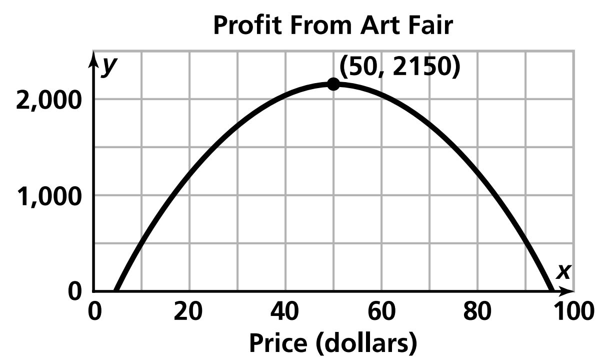
b. Possible answers: This equation is not equivalent to the other two because its graph is different. Or, substituting the same value for p into all three equations proves that they are not equivalent. For example, substituting 20 for p gives the following values for I: $I = (100 - p)p = (100 - 20)\,20 = (80)\,20 = 1{,}600$.
$I = 100p - p^2 = 100(20) - 20^2 = 2000 - 400 = 1{,}600$.
$I = 100 - p^2 = 100 - 20^2 = 100 - 400 = -300$.

c. $M = (100 - p)p - 350$, or $M = 100p - p^2 - 350$.

d. A price of \$50 gives the maximum profit, which is \$2,150. Note: This can be seen in a graph or a table of the equation as shown below.

Profit From Art Fair

Price (\$)	Profit (\$)
10	550
20	1,250
30	1,750
40	2,050
50	2,150
60	2,050
70	1,750
80	1,250
90	550



e. For prices under about \$3.65 and over about \$96.35, the potter will lose money, so the potter will make a profit on prices between these amounts. (Note: These points are the x-intercepts; students can approximate them by making a table or a graph.)

33. a. $A = x^2$; $P = 4x$

b. $A = (2x)^2 = 4x^2$, so the area would increase by a factor of 4. $P = 4(2x) = 8x$, so the perimeter would increase by a factor of 2. (Note: Students may solve this by testing several examples.)

c. $A = (3x)^2 = 9x^2$, so the area would increase by a factor of 9. Since $P = 4(3x) = 12x$, the perimeter would increase by a factor of 3.

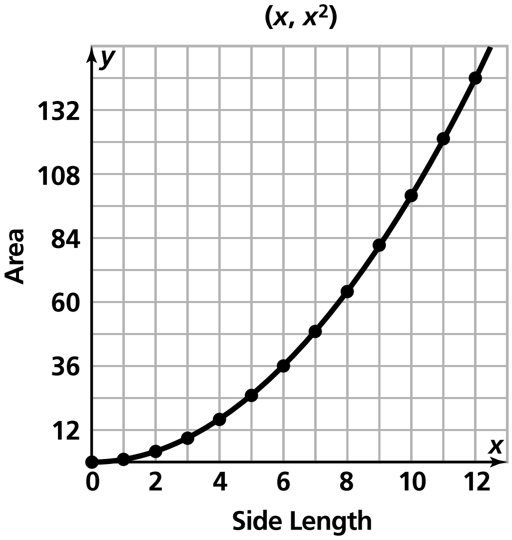
d. Since $A = 36\ m^2$, $x = 6$ m, so $P = 4(6) = 24$ m.

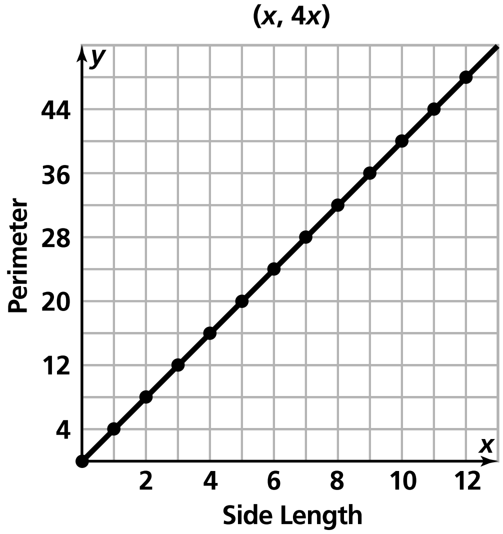
e.

Side Length, Perimeter and Area of a Square

x	$4x$	x^2
0	0	0
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25
6	24	36
7	28	49
8	32	64
9	36	81
10	40	100
11	44	121
12	48	144

f.





g. The relationship is quadratic between the side length (x) and the area (x^2). The relationship is linear between the side length (x) and the perimeter ($4x$).

34. a. $12 \times 12 = 144$ eggs in each layer.

b. $144 \times 12 = 1{,}728$ eggs in the container.

35. a. $V = x^3$

b. $V = (2x)^3 = 8x^3$; the volume would increase by a factor of 8.

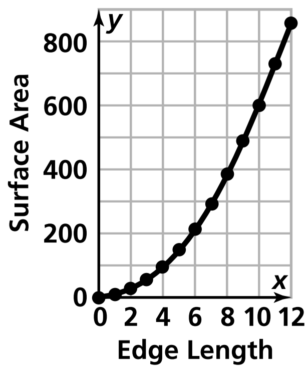
c. $V = (3x)^3 = 27x^3$; the volume would increase by a factor of 27 and the surface area would increase by a factor of 9.

d.

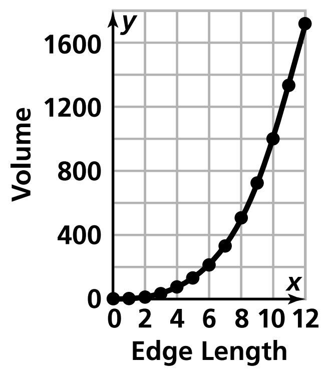
Length, Surface Area and Volume of a Solid

Edge Length	Surface Area	Volume
0	0	0
1	6	1
2	24	8
3	54	27
4	96	64
5	150	125
6	216	216
7	294	343
8	384	512
9	486	729
10	600	1,000
11	726	1,331
12	864	1,728

e. **Edge Length vs. Surface Area**



Edge Length vs. Volume



f. The relationship between edge length and surface area appears to be quadratic. The graph looks quadratic, and second differences are constant. (Figure 8)

The relationship between edge length and volume appears to be some new type of relationship because it is not a linear, quadratic, or exponential relation. (Figure 9)

36. a. $-3x(2x - 1) = -6x^2 + 3x$

b. $1.5x(6 - 2x) = 9x - 3x^2$

37. a. $(x + 1)(x + 1) = x^2 + 2x + 1$

b. $(x + 5)(x + 5) = x^2 + 10x + 25$

c. $(x - 5)(x - 5) = x^2 - 10x + 25$

The pattern is squaring a binomial, $(x + c)^2$ when the coefficient of x is 1. The square of a binomial is the square of x plus $2(c)(x)$ plus the square of c. Symbolically this is represented by: $(x + c)^2 = (x + c)(x + c) = x^2 + cx + cx + c^2$ or $x^2 + 2cx + c^2$. A similar pattern holds when the coefficient of x is not 1. $(ax + c)^2 = (ax + c)(ax + c) = (ax)^2 + acx + acx + c^2$ or $(ax)^2 + 2acx + c^2$.

Figure 8

Edge Length (cm)	Surface Area (cm²)
0	0
1	6
2	24
3	54
4	96
5	150
6	216

First Differences	Second Differences
6	12
18	12
30	12
42	12
54	12
66	

Figure 9

Edge Length (cm)	Volume (cm³)
0	0
1	1
2	8
3	27
4	64
5	125
6	216

First Differences	Second Differences	Third Differences
1	6	6
7	12	6
19	18	6
37	24	6
61	30	
91		

38. **a.** $(x + 1)(x - 1) = x^2 - 1$

b. $(x + 5)(x - 5) = x^2 - 25$

c. $(x + 1.5)(x - 1.5) = x^2 - 2.25$

The pattern is multiplying the sum and difference of two numbers. The result is the difference of the squares of the two numbers. Symbolically, this is represented by:

$(x + c)(x - c) =$
$x^2 + cx - cx - c^2$ or $x^2 - c^2$. A similar pattern holds when the coefficient of x is not 1. $(ax + c)(ax - c) = (ax)^2 - c^2$.

39. **a.** $x^2 + 6x + 9 = (x + 3)^2$

b. $x^2 - 6x + 9 = (x - 3)^2$

c. $x^2 - 9 = (x + 3)(x - 3)$

d. $x^2 - 16 = (x + 4)(x - 4)$

40. **a.** $2x^2 + 5x + 3 = (2x + 3)(x + 1)$

b. $4x^2 - 9 = (2x + 3)(2x - 3)$

c. $4x^2 + 12x + 9 = (2x + 3)(2x + 3)$

41. **a.** The areas are π square units and 4π square cm.

b. The relationship is quadratic. The area increases by increasing amounts. Students might examine the differences in areas, or they might graph the radii and area to see if they get a quadratic, or they might use symbols to justify that $y = 3.14x^2$ is a quadratic relationship. (Figure 10)

c. The length of the smaller rectangle is the same as the circumference of the smaller circle or 2π. So the surface area of the smaller cylinder is $\pi + \pi + (2\pi)(2)$, or 6π square units. The surface area of the larger cylinder is $4\pi + 4\pi + 4\pi(2)$ or 16π square units.

d. Yes; students might examine second differences and see that they are a constant 2π. Or they might identify the equation $y = 2\pi x(x + 2)$ as the equation of a parabola with x-intercepts at 0 and -2. (Figure 11)

42. B

43. **a.** (Figure 12)

b. Yes; students might examine the second differences and see that they are a constant 8π or they might identify the relationship's equation of $y = 4\pi x^2$ as the equation of a parabola.

44. **a.** Each edge is 3 units.

b. The surface area is 54 square units. The volume is 27 cubic units.

c. Student drawings should show the flat pattern of a cube with edge 4 units, surface area 6(16) or 96 square units.

Figure 10

Relationship of a Radius to Area of a Circle

Radius	1	2	3	4	x
Area	π	4π	9π	16π	πx^2

Figure 11

Surface Areas of Cylinders With Different Radius and Height

Radius	1	2	3	4	x
Height	2	2	2	2	2
Surface Area	6π	16π	$[9 + 9 + (6)(2)]\pi = 30\pi$	$[16 + 16 + (8)(2)]\pi = 48\pi$	$\pi x^2 + \pi x^2 + (2\pi x)(2) = 2\pi x^2 + 4\pi x = 2\pi x(x + 2)$

Figure 12

Surface Areas of Cylinders With Equal Radius and Height

Radius	1	2	3	4	x
Height	1	2	3	4	x
Surface Area	$\pi + \pi + (2\pi)(1) = 4\pi$	$4\pi + 4\pi + (4\pi)(2) = 16\pi$	$9\pi + 9\pi + (6\pi)(3) = 36\pi$	$16\pi + 16\pi + (8\pi)(4) = 64\pi$	$\pi x^2 + \pi x^2 + (2\pi x)(x) = 4\pi x^2$

ACE ANSWERS 4

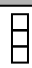
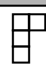
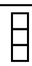
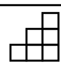
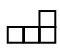
d. $V = x^3$. This is not quadratic (it is actually a cubic relationship). Students might make a table and examine how the volume grows, or they might graph $y = x^3$ and examine the shape, or they might refer to the symbols.

45. No; the surface area of Silvio's box is 1,536 sq. in, since $16^2 \times 6 = 1{,}536$. Ten sq. ft. of wrapping paper equals 1,440 sq. in since a square foot is 144 square inches and $10(144 \text{ in}) = 1{,}440$ sq. inches. There will not be enough paper.

46. H

47. C

48.

Building	1	2	3	4
Base				
Front				
Right				

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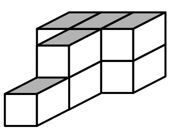

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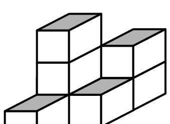
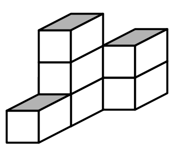


Building 1



Building 2

Extensions

51.a. If only the 20 soccer team members go, the cost of the trip is \$125 per student. The travel agent's profit is the difference between income and cost, or $P = 125n - 75n$, where n is the number of students: $P = 125(20) - 75(20) = 2{,}500 - 1{,}500 = \$1{,}000$.

b. If 25 students go, the cost is \$120 per student and the agent's profit is $P = 120n - 75n = 120(25) - 75(25) = 3{,}000 - 1{,}875 = \$1{,}125$.

c. If 60 students go, the cost is \$85 per student and the agent's profit is $P = 85n - 75n = 85(60) - 75(60) = 5{,}100 - 4{,}500 = \$600$.

d. If 80 students go, the cost is \$65 per student and the agent's profit is $P = 65n - 75n = 65(80) - 75(80) = 5{,}200 - 6{,}000 = -\$800$. For this many students, the travel agent would lose money.

52. (Figure 13)

a. price per student $= 125 - (n - 20)$, or $125 - n + 20$, or $145 - n$

b. income $=$ price $\times\, n = [125 - (n - 20)]n$, or $125n - n(n - 20)$, or $125n - n^2 + 20n$, or $145n - n^2$

c. expenses $= 75n$

d. profit $=$ income $-$ expenses $= [125 - (n - 20)]n - 75n$, or $125n - n(n - 20) - 75n$, or $-n^2 + 20n - 75n$, or $70n - n^2$.

Figure 13

Pricing and Profit Scenarios for a Travel Agent

Number of Students	Price per Student	Travel Agent's Income	Travel Agent's Expenses	Travel Agent's Profit
20	125	20 × 125 = 2,500	20 × 75 = 1,500	2,500 − 1,500 = 1,000
21	124	21 × 124 = 2,604	21 × 75 = 1,575	2,604 − 1,575 = 1,029
22	123	22 × 123 = 2,706	22 × 75 = 1,650	2,706 − 1,650 = 1,056
23	122	23 × 122 = 2,806	23 × 75 = 1,725	2,806 − 1,725 = 1,081

53. **a.** The agent's profit is greatest for 35 students.

b. If fewer than 70 students go on the trip, the agent will make a profit.

c. From 30 students to 40 students give the travel agent a profit of at least $1,200.

54. **a.** It takes 3 moves to solve the puzzle with 1 pair of coins. Starting with

1		5

the moves could be as follows:

	1	5

5	1	

5		1

It takes 8 moves to solve the puzzle with 2 pairs of coins. Starting with

1	1		5	5

the moves could be as follows:

1		1	5	5

1	5	1		5

1	5	1	5	

1	5		5	1

	5	1	5	1

5		1	5	1

5	5	1		1

5	5		1	1

It takes 15 moves to solve the puzzle with 3 pairs of coins. Starting with

1	1	1		5	5	5

The moves could be as follows:

1	1	1	5		5	5

1	1		5	1	5	5

1		1	5	1	5	5

1	5	1		1	5	5

1	5	1	5	1		5

1	5	1	5	1	5	

1	5	1	5		5	1

1	5		5	1	5	1

	5	1	5	1	5	1

5		1	5	1	5	1

5	5	1		1	5	1

5	5	1	5	1		1

5	5	1	5		1	1

5	5		5	1	1	1

5	5	5		1	1	1

b. (Figure 15)

c. The numbers of moves calculated from the expression agree with the numbers found above. (Figure 14)

d. Second differences are a constant 2, so the relationship is quadratic. (Figure 15)

55. a. The graph of $y_1 = x + 1$ is a straight line with slope 1 and y-intercept $(0, 1)$. The graph of $y_2 = (x + 1)(x + 2)$ is a parabola with a minimum point at $(-1.5, -0.25)$ and x-intercepts at $(-1, 0)$ and $(-2, 0)$. The graph of $y_3 = (x + 1)(x + 2)(x + 3)$ increases as x increases, then decreases, then increases again. It has three x-intercepts at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$. The graph of $y_4 = (x + 1)(x + 2)(x + 3)(x + 4)$ is shaped like the letter W. It has two local minimum points, a local maximum point, and four x-intercepts at $(-1, 0)$, $(-2, 0)$, $(-3, 0)$, $(-4, 0)$. Note to teacher: The terms *local minimum* and *local maximum* will be introduced in future mathematics courses. They just refer to minimums and maximums over a given part of the graph, which are not necessarily the minimum or maximum for the entire graph.

b. The equation $y_1 = (x + 1)$ has constant first differences. The equation $y_2 = (x + 1)(x + 2)$ has constant second differences. The equation $y_3 = (x + 1)(x + 2)(x + 3)$ has constant third differences. The equation $y_4 = (x + 1)(x + 2)(x + 3)(x + 4)$ has constant fourth differences.

56. a. blue: 4
yellow: $4 \times 3 = 12$
orange: $3 \times 3 = 9$

b. blue: 4
yellow: $4 \times 8 = 32$
orange: $8 \times 8 = 64$

c. blue: 4
yellow: $4(n - 2)$
orange: $(n - 2) \times (n - 2)$ or $(n - 2)^2$

d. The relationship described by $(n - 2)^2$ is quadratic because it is formed by the product of two linear factors.

57. a. 0 cubes

b. 6 cubes

c. 16 cubes

d. 8 cubes

e. $8 + 16 + 6 = 30$ cubes, or $3 \times 2 \times 5 = 30$ cubes

Figure 14

# of Type of Coin	1	2	3	4	5	6	7	8	9	10
# of Moves	3	8	15	24	35	48	63	80	99	120

Figure 15

Number of Each Type of Coin	Number of Moves
1	3
2	8
3	15
4	24
5	35
6	48
7	63
8	80
9	99
10	120

First Differences
5
7
9
11
13
15
17
19
21

Second Differences
2
2
2
2
2
2
2
2

Possible Answers to Mathematical Reflections

1. Possible situations: (i) the nth triangular number. Question: What is the 20th triangular number? (ii) the height in a frog jump. Question: What is the highest height in a frog jump? (iii) the number of high fives. Question: How many high fives are there if everyone exchanges high fives with each other on a team with 10 members? (iv) rectangles with a fixed perimeter. Question: What is the greatest area for a rectangle with a fixed perimeter of 60 meters?
2. a. In tables of (x, y) values for quadratics the first differences are non-constant but the second differences are constant.

   b. The graphs of quadratics (if a view window including all four quadrants is used and if a big enough range of values for the x- and the y-values is used) are parabolas opening upward or downward depending on the sign of the coefficient of the x^2 term. The graph has a line of symmetry through the maximum or minimum point. The line of symmetry intersects the x-axis at the midpoint between the x-intercepts.

   c. The equations that match quadratic relations can be in expanded form or factored form. In the expanded form, $y = ax^2 + bx + c$, the highest exponent of the independent variable is 2. If a is positive, then there is a minimum point; if a is negative, then there is a maximum point. The y-intercept is c. In a factored form, $y = (x + b)(x + c)$, there are two factors, each of which has the independent variable raised to the first power. The x-intercepts are $x = -b$ and $x = -c$.
3. The patterns of change for *linear* functions are characterized by constant first differences. The patterns of change for *exponential* functions are characterized by consecutive differences either increasing without bound or approaching 0 but never achieving it. The patterns of change for *quadratic* functions are characterized by a constant second difference. Note to the teacher: These are only general trends and simply because consecutive differences increase without bound does not imply that a function is exponential.

Answers to Looking Back and Looking Ahead

1. a. All graphs will be parabolas that are concave down with the y-intercept 0 and x-intercepts at 0 and $\frac{b}{16}$ and maximum point $(\frac{b}{32}, \frac{b^2}{64})$. At this point in the students' learning about quadratic functions and equations we don't expect such complete abstract reasoning. Most students should know the general shape of the graphs and recognize in general that the maximum point will lie midway between the two x-intercepts.

   b. Table 1 was produced by the necessary quadratic, while Table 2 shows a constant rate of rise and fall in height, and thus cannot be quadratic. Students might notice that if one looks at the first and second differences of height values in Table 1, you get 5, 3, 1, −1, −3, −5, and then −2, −2, −2, −2, −2. Constant second differences are another indicator of a quadratic relationship.
2. a. 16 feet. This answer can be found by tracing a table or graph of the height equation. Symbolic reasoning might be used to find that the ball comes back to its starting position when $t = 2$ and to infer that the maximum point occurs midway in that time interval.

   b. 2 seconds. Students might find this result by tracing a table or graph of the equation or by solving with symbolic reasoning.

   $$-16t^2 + 32t = 0$$
   $$-16t(t - 2) = 0$$
   $$t = 0 \text{ and } t = 2$$

   c. 0.5 seconds and 1.5 seconds. Again, students might answer this question by tracing a table or graph of the equation to find x-values for which $y = 12$. We do not really expect students to solve the equation $-16t^2 + 32t = 12$ by symbolic reasoning.

3. a. width $= 90 - x$ or width $= \frac{180 - 2x}{2}$

   b. $A = x(90 - x)$

   c. $x(90 - x) = 2{,}000$; so $x = 40$ and $w = 50$ or $x = 50$ and $w = 40$
   Students will probably solve the equation by tracing a table or graph of the equation $y = x(90 - x)$ to find values of x that give an area of 2,000, not by formal reasoning with the symbols alone.

   d. $x = w = 45$ gives a maximum area of 2,025 square feet. Students could use a calculator to find the x-value that corresponded to the largest y-value in the table. They could use the trace function to approximate the maximum point on the graph of the parabola or they could use the maximum function to find the maximum point on the parabola. Students might also reason that the figure with maximum area and fixed perimeter is a square, so each side would be $\frac{180}{4} = 45$ feet.

4. a. As x goes up by a constant value, the y-values either rise to a maximum and then decrease in a symmetric pattern or fall to a minimum and then increase in a symmetric pattern. The rate of change in y-values is not constant, but the second differences of y-values will be constant for a sequence of equally spaced x-values. In general, the rate of change in y-values is smallest near the maximum or minimum point and increases as x-values move away from that point.

   b. Graphs of all quadratics are parabolas that are symmetric about the vertical line through their maximum or minimum point. The parabolas are concave up (open upward) if the coefficient of the x^2 term is positive and concave down if that coefficient is negative. The y-intercept of any quadratic graph is the constant term in the equation for that graph.

   c. Any quadratic relation can be expressed with an equation in the form $y = ax^2 + bx + c$, though other equivalent forms are often useful. In particular, the factored form of $y = a(x - r)(x - s)$ can be used to find the x-intercepts $(r, 0)$ and $(s, 0)$ by inspection.

5. a. Differences include the following ideas: Equations of linear relations have exponents equal to one; the tables of linear relations show a constant rate of change in output values (y) as input values (x) change (either always increasing or always decreasing); and the graphs of linear relations are straight lines.

   b. Equations of exponential relations are of the form $y = a(b^x)$; in tables of exponential relations $y = a(b^x)$ any increase of 1 in x corresponds to changes in y by the factor b (always increasing or always decreasing); and the graphs of exponential relations are curves that resemble half of a parabola. They do not have symmetry about a vertical line as parabolas do.

6. If a is positive the parabola will have a minimum point. If a is negative, the parabola will have a maximum point.

7. a. Find the value of x when $y = 0$ by tracing in a table.

   b. Find the x-intercepts of the function graph.