Investigation 3

ACE Assignment Choices

Differentiated Instruction
Solutions for All Learners

Problem 3.1

Core 1–3, 20, 21, 23–25
Other *Applications* 4–8; *Connections* 22, 26–28; *Extensions* 47, 48; unassigned choices from previous problems

Problem 3.2

Core 9–11, 29–31
Other *Connections* 32–37; *Extensions* 49; unassigned choices from previous problems

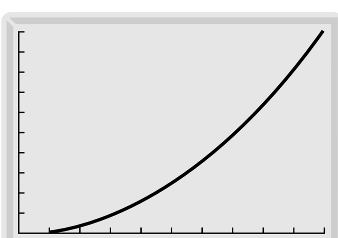
Problem 3.3

Core 12–16, 41–45
Other *Applications* 17–19; *Connections* 38–40, 46; *Extensions* 50, 51; unassigned choices from previous problems

Adapted For suggestions about adapting Exercise 9 and other ACE exercises, see the CMP *Special Needs Handbook*.
Connecting to Prior Units 37: *Shapes and Designs*; 39, 40: *Filling and Wrapping*

Applications

1. a. 25 and 36
   b. n^2
   c. The numbers seem to be getting bigger by a larger amount each time. The square number increases by consecutive odd integers, beginning with 3: 3, 5, 7,. . . .



X	Y1	
0	0	
1	1	
2	4	
3	9	
4	16	
5	25	
6	36	
X=0		

X	Y1	
6	36	
7	49	
8	64	
9	81	
10	100	
11	121	
12	144	
X=12		

2. a. 2, 6, 12, 20, 30, 42
   b. The rectangular number increases by consecutive even integers beginning with 4: 4, 6, 8, 10, 12,. . . .
   c. The seventh number is 14 greater than the sixth number, or 56. The eighth number is 16 greater than the seventh number, or 72.
   d. $r = n(n + 1)$, where r is the rectangular number.
3. a. $\frac{18(18 + 1)}{2} = \frac{18(19)}{2} = 171$
   b. yes; It is a triangular number because $\frac{20(20 + 1)}{2} = \frac{20(21)}{2} = 210$ (Note: students may substitute into the equation or continue the table to answer this.)
4. a. Sam is correct because if you look at the triangles in Problem 3.1, each row of a triangle represents a counting number: 1, 2, 3 . . . . Therefore the equation for triangular numbers, $\frac{n(n + 1)}{2}$, represents the sum of the numbers 1 to n.
   b. 55 c. 120 d. $\frac{n(n + 1)}{2}$

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5. Rectangular, because it satisfies the equation $r = n(n + 1)$, 110 is the tenth rectangular number. $110 = 10(11)$

6. Triangular, because it satisfies the equation $t = \frac{n(n + 1)}{2}$, 66 is the 11th triangular number: $66 = \frac{11(12)}{2}$.

7. Square, because it satisfies the equation $s = n^2$, 121 is the 11th square number: $121 = 11^2$.

8. None, because it does not satisfy any of the associated equations, 60 is none of these.

9. **a.** The eight people on each side shake hands with 8 others, so $8^2 = 64$ handshakes will be exchanged.
   b. 28. $7 + 6 + 5 + 4 + 3 + 2 + 1 = \frac{7(8)}{2} = 28$, or each of 8 people shake hands with the other 7, but as this counts each handshake twice, $\frac{8(7)}{2} = 28$ handshakes will be exchanged.
   c. 12. $6 \times 2 = 12$

10. **a.** 15. $5 \times 3 = 15$
   b. 10. $4 + 3 + 2 + 1 = \frac{4(5)}{2} = 10$, or each of 5 people high-five with the other 4, but as this counts each high five twice, $\frac{4(5)}{2} = 10$ high fives will be exchanged.

11. **a.** 10. $4 + 3 + 2 + 1 = \frac{4(5)}{2} = 10$, or each of 5 rooms connects with each other 4 by cables, but as this counts two cables between each two rooms, $\frac{4(5)}{2} = 10$ cables will be needed.
   b. 21. $6 + 5 + 4 + 3 + 2 + 1 = \frac{6(7)}{2} = 21$.
   c. They are the same situation mathematically, where the cables are associated with high fives and rooms are associated with people.

12. Possible answer: P represents the area of a rectangle formed from a square of a side length n in which one dimension is decreased by 1 unit, or P represents the number of handshakes between a team of n players and a team of $n - 1$ players.

13. Possible answer: P represents the area of a rectangle with sides of length 2 and n.

14. Possible answer: P represents the area of a rectangle formed from a square of a side length n in which one dimension is decreased by 2 units.

15. Possible answer: P represents the area of a rectangle with perimeter 32 units.

16. **a.** Graph ii; Possible explanation: The equation $h = \frac{n}{2}(n - 1)$, which represents the relationship between the number of high fives and the number of team players, tells us that the graph will have x-intercepts of 0 and 1.
   b. Graph iii; Possible explanation: The area of rectangles with a fixed perimeter grows and then declines as the length of a side increases. That means the graph has a maximum point.
   c. Graph i; possible explanation: Since the equation for the relationship described is $y = (x + 2)(x - 3)$, we know that $y = 0$ when $x = -2$ or 3. So, the x-intercepts must be -2 and 3.
   d. Graph iv; Possible explanation: The nth triangular number can be represented by the equation $T = \frac{n(n + 1)}{2}$. This equation that tells us the graph will have two x-intercepts of 0 and -1.

17. **Quadratic Relationship 1**

x	y
0	0
1	1
2	3
3	6
4	10
5	15
6	21

18. **Quadratic Relationship 2**

x	y
0	0
1	3
2	8
3	15
4	24
5	35
6	48

19. Quadratic Relationship 3

x	y
0	0
1	4
2	6
3	6
4	4
5	0
6	−6

Connections

20. a.

4	$4x$	8
x	x^2	$2x$
	x	2

1	$1x$	8
x	x^2	$8x$
	x	8

Expanded form: $x^2 + 6x + 8$; factored form: $(x + 2)(x + 4)$ and expanded form: $x^2 + 9x + 8$; Factored form: $(x + 8)(x + 1)$. We can't have negative values for sides given the area models in the student edition so factored forms like $(x - 2)(x - 4)$ aren't possible even though they give you the terms 8 and x^2.

b. No; expanded form: $x^2 + 6x + 5$; Factored form: $(x + 1)(x + 5)$ (excluding commutation of the factors of $x + 1$ and $(x + 5)$. Again there are more possibilities if we use non-whole-number factors of 5.

21. $2x^2 + 3x + 3x + 3x + 9 = 2x^2 + 9x + 9$ or $(2x + 3)(x + 3)$

22. a. $2x^2 + 7x + 6$

	x	2
$2x$	$2x^2$	$4x$
3	$3x$	6

$x^2 + 6x + 8$

	x	4
x	x^2	$4x$
2	$2x$	8

b. $2x^2 + 7x + 6 = (2x + 3)(x + 2)$; $x^2 + 6x + 8 = (x + 4)(x + 2)$; $x^2 + 6x + 8$ is easier to do because x^2 has a coefficient of one.

23. $x(5 - x) = 5x - x^2$

24. $(x + 1)(x + 3) = x^2 + 4x + 3$

25. $(x - 1)(x + 3) = x^2 + 2x - 3$

26. $3x(x + 5) = 3x^2 + 15x$

27. $(2x + 1)(x + 3) = 2x^2 + 7x + 3$

28. $(2x - 1)(x + 3) = 2x^2 + 5x - 3$

29. $x^2 - 9x + 8 = (x - 8)(x - 1)$

30. $4x^2 - 6x = 2x(2x - 3)$

31. $x^2 - 2x - 3 = (x - 3)(x + 1)$

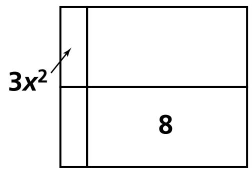
32. $3x^2 + 14x + 8 = (3x + 2)(x + 4)$

33. $4x^2 + 6x = 2x(2x + 3)$

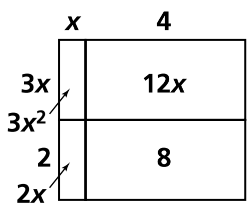
34. $4x^2 - x - 3 = (4x + 3)(x - 1)$

35. $x^3 - 2x^2 - 3x = x(x + 1)(x - 3)$

36. a. Subdivide a rectangle into four parts. Label the area of one of the smaller rectangles as $3x^2$ and the one diagonal to it as 8. Use these areas to find the dimensions of these two smaller rectangles. (Note that there are several ways to do this.) Once you have picked the dimensions, use them to find the area of the remaining two rectangles. If the sum of the area of these two rectangles is $14x$, then you picked the right dimensions. And the dimensions that you picked for the first two rectangles are the dimensions of the original rectangle. These two dimensions are the factors in the factored form of $3x^2 + 14x + 8$.



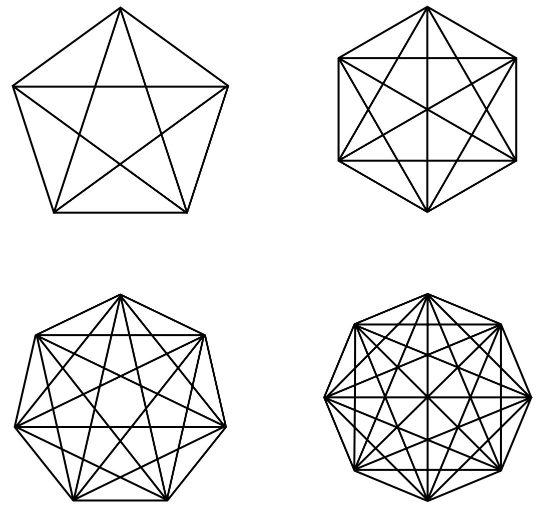
Let the dimensions of the rectangle whose area is $3x^2$ be $3x$ and x and the dimensions of the rectangle with area 8 be 4 and 2. The areas of the remaining two rectangles are $2x$ and $12x$ and their sum is $14x$.



The dimensions of the original rectangle are $3x + 2$ and $x + 4$. Then write $14x$ as $12x + 2x$. Label the area of the two remaining rectangles as $12x$ and $2x$.

b. Look at the factors of the coefficient of x^2 and the factors of the constant term. Put these values in the factor pairs: $(\underline{\ ?\ }x + \underline{\ ?\ })(\underline{\ ?\ }x + \underline{\ ?\ })$. Use the distributive property to check if the coefficient of x is correct.

37. a. A pentagon has 5 diagonals, a hexagon has 9 diagonals, a heptagon has 14 diagonals and an octagon has 20 diagonals.



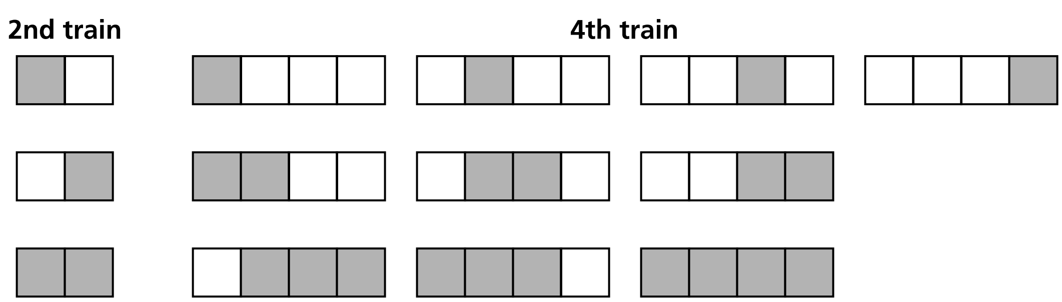
b. An n-sided polygon has $\frac{n(n-3)}{2}$ diagonals.

This problem could be solved like the high fives problem. Each of n points exchanges high fives with the other $n - 3$ points (diagonals) excluding the adjacent points and itself but as this counts each high five twice, $\frac{n(n-3)}{2}$ high fives will be exchanged.

38. a. The first train has 1 rectangle, the square itself. The second train has 3 rectangles, as shown below. The third train has 6 rectangles, as in the problem. The fourth train has 10 rectangles, as shown below. The fifth train has 15 rectangles (5 one-square rectangles, 4 two-squares rectangles, 3 three-square rectangles, 2 four-square rectangles and 1 five-square rectangle.)

(Figure 8)

Figure 8



b. **Number of Rectangles in the First Ten Trains**

Train	Number of Rectangles
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55

c. The number of rectangles increases by a greater amount each time. The pattern of increase is 2, 3, 4, 5, 6, ... So, we could expect an increase of 15 from the 14th train to the 15th train. The table below shows that there are 120 rectangles in the 15th train.

Number of Rectangles in Trains 11 Through 15

Train	Number of Rectangles
11	66
12	78
13	91
14	105
15	120

d. $r = \frac{n(n+1)}{2}$ where r is the number of rectangles (Note: The numbers are the same as the triangular numbers, an observation that students may use to solve this equation).

e. $r = \frac{15(15+1)}{2} = \frac{15(16)}{2} = \frac{240}{2} =$ 120 rectangles.

39 **a.** About 78.5 cm^2

b. About 78.5 centimeter cubes

c. 10 layers

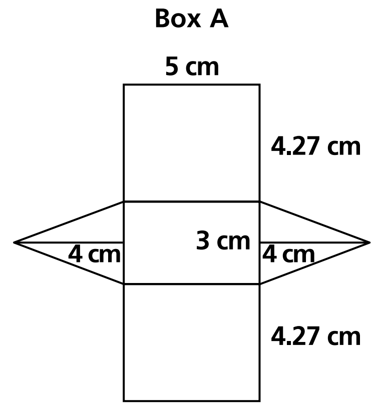
d. $\approx$ 785 cm^3; this is the product of the number of cubes in one layer and the number of layers that fill the can.

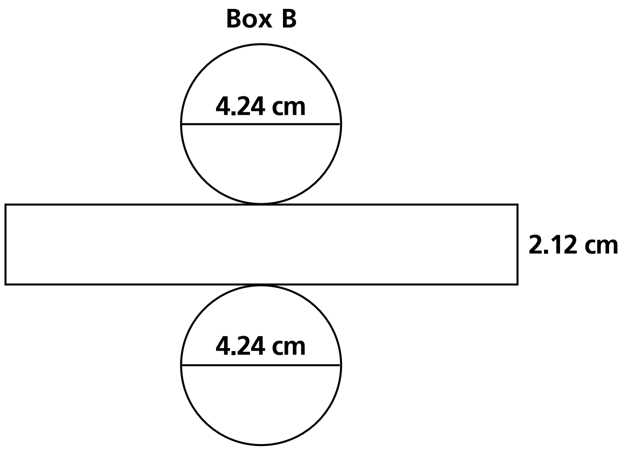
e. $\approx$ 31.4 cm

f. $\approx$ 314 cm^2

g. $\approx$ 471 cm^2; this is the sum of twice the area of the base and the area of the paper label.

40. **a.** Possible net:





b. Box A: 69.7 cm^2, Box B: 56.48 cm^2; To find the surface area of box A you would have to add the area of the base which is $3 \times 5 = 15$ cm^2, the area of the two triangles each of which has an area of $\frac{1}{2}(3 \times 4) = 6$ cm^2 and the areas for the side rectangles. The rectangles have dimensions 5 and about 4.27. To find the 4.27 you use the Pythagorean theorem on the right triangle on the front of the box with the side lengths of 3 and 4 to obtain the hypotenuse of $\sqrt{1.5^2 + 4^2} = 4.27$. So the surface area is $15 + 2(6) + 2(5 \times 4.27) = 69.7$ cm^2. To find the surface area of Box B, you would have to add the area of the top and bottom two circles, which each have an area of $\pi(2.12)^2 \approx 14.12$ cm^2, to the area of the

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rectangle which has an area of $2.12 \times \pi(4.24) \approx 28.24$ cm^2. So the surface area is $14.12 \times 2 + 28.24 = 56.48$ cm^2.

c. Box A will require more cardboard to construct, since it has a larger surface area.

41. None of the above.

42. Quadratic

43. Exponential

44. None of the above

45. D

46. H; one way to eliminate certain choices is to notice that H is the only parabola that can have a minimum point since the other three parabolas open down. You can see this by looking at the coefficient of x^2.

Extensions

47. a. $(1 + 100) + (2 + 99) + \ldots + (99 + 2) + (100 + 1) = \frac{100}{2} \times (101) = 50 \times 101 = 5{,}050$

b. This idea could be represented by the equation $s = \frac{n}{2}(n + 1)$, where s is the sum of the first s whole numbers.

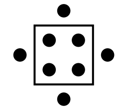
c. This method is the same as Gauss's method in the Did You Know? box. It just pairs the numbers in a number sentence rather than drawing arrows to make the pairings.

48. a. 8, 21, 40

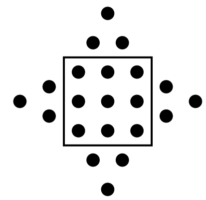
b. 65, 96, 133; $5 \times 5 + 10 \times 4 = 65$; $6 \times 6 + 15 \times 4 = 96$; $7 \times 7 + 21 \times 4 = 133$.

c. $s = (n + 1)^2 + \frac{4n(n + 1)}{2}$

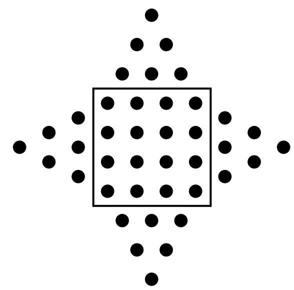
For the Teacher: Students may not be able to find this equation. You might help them to write an equation by pointing out that each star has a center square and four triangular points. Each star number is thus composed of 4 times the triangular number of the same number plus the next square number. For example, the first star number has a center of 4 (the second square number) and four "points" of 1 (the first triangular number); the second star number has a center of 9 (the third square number) and four points of 3 (the second triangular number).



1st

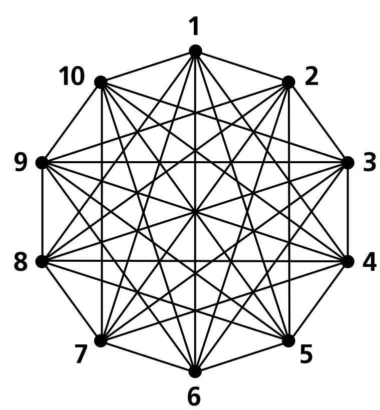


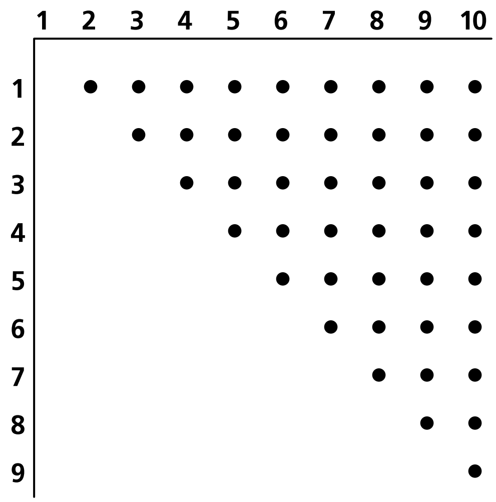
2nd



3rd

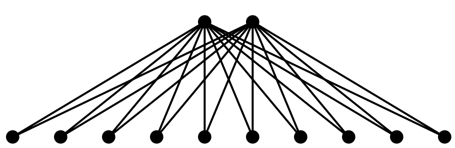
49. a. 45.; $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = \frac{9(10)}{2} = 45$, or each of the 10 classmates shakes hands with 9 others, but as this counts each handshake twice, $\frac{10(9)}{2} = 45$ handshakes exchanged. Three diagrams that express this are shown below.

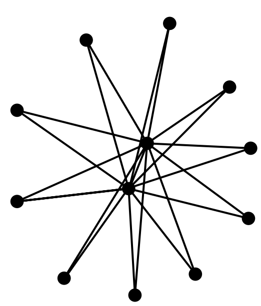


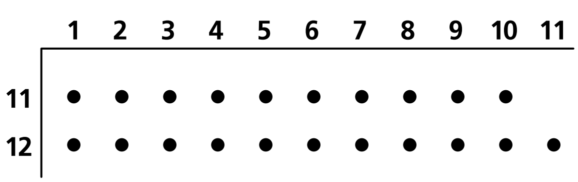


People: A, B, C, D, E, F, G, H, I, J
Handshakes: AB, AC, AD, AE, AF, AG, AH, AI, AJ
BC,BD, BE, BF, BG, BH, BI, BJ
CD, CE, CF, CG, CH, CI, CJ
DE, DF, DG, DH, DI, DJ
EF, EG, EH, EI, EJ
FG, FH, FI, FJ
GH, GI, GJ
HI, HJ
IJ

b. Each of the 2 friends shakes hands with 11 others, but as one handshake is counted twice (when they shake hands with each other), there are $2(11) - 1 = 21$ handshakes in all. Three diagrams that express this are shown here.





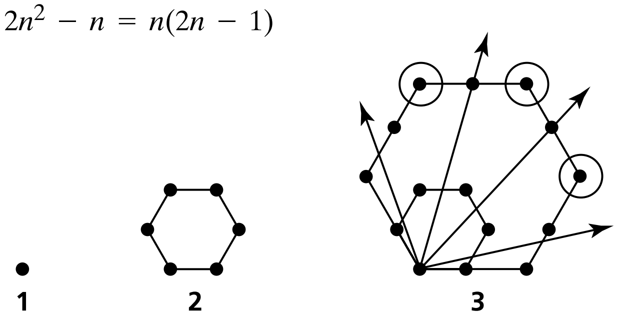


50. a. 1, 6, 15

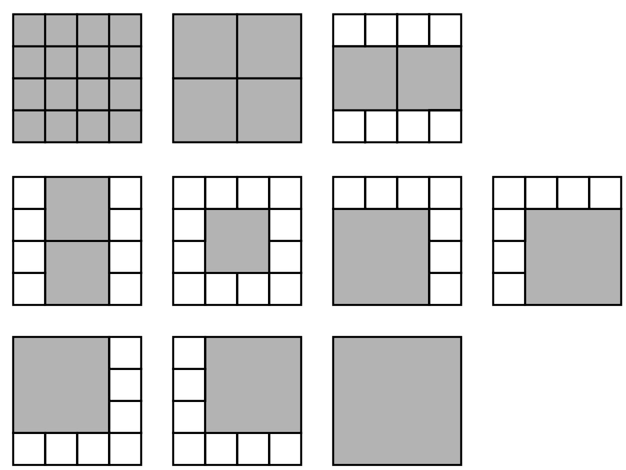
b. 28, 45

c. $h = n(2n - 1)$.

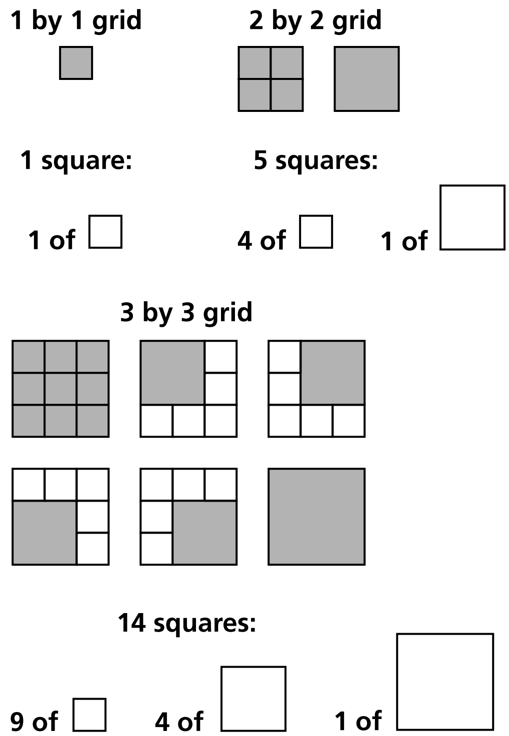
For the teacher: Students may not be able to find this equation. You might help them write an equation by pointing out that there are four lines along which they can add $1 + 2 + 3 + \ldots + n$ dots. The equation for triangular numbers, $\frac{n(n + 1)}{2}$, is multiplied by 4 to get this sum. However, this counts the dots circled on the diagram an extra 3 times for each hexagon, for a total of $3n$ times for the whole figure. Thus, the complete equation for hexagonal numbers is as follows: $h = 4(\frac{n(n + 1)}{2}) - 3n = 2n^2 + 2n - 3n = 2n^2 - n = n(2n - 1)$



51. a. Of the 14 remaining squares, 9 are 2-by-2 squares, 4 are 3-by-3 squares, and 1 is a 4-by-4 square.



b. Possible answer: The squares found in a 1-by-1 grid, a 2-by-2 grid and a 3-by-3 grid are shown below. Each grid contains n^2 more squares than the previous grid, so an equation for the number of squares in an n-by-n grid is $s = n^2 + (n - 1)^2 + \ldots + 1^2$, where s is the number of squares.



This shows that the pattern is a sum of squares.

Possible Answers to Mathematical Reflections

1. **a.** The relationships in the handshake and triangular number problems have similar graphs and tables in which y increases at an increasing rate. The high-five situation and the triangular number pattern are applications of the same formula. However, one formula is $\frac{n}{2}(n + 1)$ and the other is $\frac{n}{2}(n - 1)$.

   b. These functions are the same as in Investigations 1 and 2, which also were expressed as the product of 2 factors. The shape of the graphs are similar—either a parabola that opens up or a parabola that opens down, with a minimum value or maximum value.
   Students might suggest things like this: In the *tables* a constant change in x does not lead to a constant change in y as it did for linear relations, nor is there an inverse relation as there was for some of the investigations in *Thinking with Mathematical Models*. The *graphs* are all parabolas when viewed in the first and second quadrants. The *equations* that represent quadratic relations all have an x^2 (or n^2) as the highest power of the input variable. However, students will not always see them in this form. They have seen expressions such as $n(n + 1)$ or $n(n - 1)$ or $6(n - 2)^2$ or $\frac{n(n + 1)}{2}$ or $\frac{n(n - 1)}{2}$. At this point they should not be expected to discuss the degree of the equation, but they should begin to notice some similarities in all of the expressions such as "an expression with n is multiplied by another expression having n in it."

2. **a.** The rate of change for a sequence of numbers associated with a quadratic equation is not constant. However, the rate of change for each increment or increase seems to be constant. For example, in the triangular numbers, 1, 3, 6, 10, 15, 21, . . . , the increases are +2, +3, +4, +5, +6, ... Each increase is one more than the last increase, and this "one more" is constant. For the handshake pattern, the increases are 3, 5, 7, 9, 11, . . . , and each increase is two more than the previous one. To predict the next entry you determine how the increments in the y-values are changing and continue the pattern. Thus, for the handshake pattern, if we know the sequence of y-values is . . . 56, 72, 90, 110, we can predict the next entry by adding 22 to get 132.

   b. If a table of values indicates a symmetric increase/decrease or decrease/increase pattern, *and* if the *change* in y-values is increasing (or decreasing) by a fixed amount, then the function is quadratic.